The Gel'fand-Tsetlin representations of the orthogonal Cayley-Klein algebras

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(Received 2 October 1991; accepted for publication 1 November 1991)

The explicit expressions of the operators of the irreducible representations of the orthogonal Cayley-Klein algebras $so_{n+1}(\mathbf{j})$ are obtained from the well-known Gel'fand-Tsetlin representations of so_{n+1} . The contractions and the analytic continuations of the representations of $so_3(j_1, j_2)$, $so_4(j_1, j_2, j_3)$ regarded as examples. This approach gives the representations of the contracted and the analytically continued algebras in different bases, for example, in discrete and continuous ones. Possible contractions of the irreducible representations are discussed.

1 Introduction

The irreducible representations of the classical orthogonal algebras $so(n+1)$ were studied more than 40 years ago by Gel'fand and Tsetlin [1]. Approximately at that time Wigner and Inonu [2] introduced the contraction of the group representations. This operation gives the representations of the contracted group from the representations of the initial group. Contraction was later used by many authors, in particular by Wigner and Kim [3], to describe the photon little group, of Smirnov and Asherova [4], in the symplectic collective model of the complex nucleus, of Celeghini, Tarlini, and Vitiello [5] in the theory of the spontaneous breakdown of symmetry, of Doebner and Melsheimer [6], to describe the limiting process connecting the dynamical group of the interacting system with the dynamical group of the corresponding free particle.

There is another approach in group theory, namely the Weyl unitary trick [7], which gives the representations of the pseudoorthogonal group starting from the classical orthogonal group. This idea was used by various authors. Particular mention must be made of the work of Dirac [8], Holman and Biedenharn [9], and Kuriyan, Mukunda, and Sudarshan [10], who have developed this trick to the method of master analytic representation.

The purpose of this paper is to present the unified description of the orthogonal Cayley-Klein algebras and their representations starting from the well-known Gel'fand-Tsetlin representations of the classical orthogonal algebras. The orthogonal Cayley-Klein algebras $so_{n+1}(\mathbf{j})$ are obtained from

the classical ones by Wigner-Inonu contractions and analytical continuations (or the Weyl unitary trick). The unified description [11]–[13] is achieved by introducing parameters, which may be equal to the real unit or to the imaginary unit or to the Clifford dual units. The main feature of our approach is the transformations of the well-known generators of the irreducible representations of so_{n+1} into the generators of therepresentations of the algebras under consideration. The results of Chakrabarti [14] concerning the representations of the inhomogeneous group (algebra) $ISO(n)$ are contained in our results as the special case of the $j_1 = \iota_1$ contraction. The Gel'fand-Tsetlin representations of the unitary Cayley-Klein algebras were studied in a previous paper [15].

This paper is organized as follows. In Sec. II we recall the unified description of the orthogonal Cayley-Klein groups and their algebras. In Sec. III we define the transformations of the components of the Gel'fand-Tsetlin patterns and obtain the general formulas for the generators of the representations of $so_{n+1}(\mathbf{j})$. Contractions and analytic continuations of the irreducible representations of the orthogonal algebras $so_3 j_1, j_2$ and $so_4(j_1, j_2, j_3)$ are regarded as examples in Sees. IV and V. Possible contractions of the irreducible representations of $so_{n+1}(\mathbf{j})$ are briefly discussed in Sec. VI.

2 The orthogonal Cayley–Klein algebras

Let us define the map of the n -dimensional Euclidean space $\mathbf{R}_{n+1}$ into the space $\mathbf{R}_{n+1}(\mathbf{j})$ as follows [11]–[13]:

$$\psi : \mathbf{R}_{n+1} \rightarrow \mathbf{R}_{n+1}(\mathbf{j})$$

$$\psi x_0^* = x_0, \psi x_k^* = x_k \prod_{m=1}^k j_m, k = 1, 2, \dots, n, \quad (1)$$

where $x_0^*, x_k \in \mathbf{R}_{n+1}$, $x_0, x_k \in \mathbf{R}_{n+1}(\mathbf{j})$ are the Cartesian coordinates, $\mathbf{j} = (j_1, j_2, \dots, j_n)$, each of the parameters $j_k = 1, \iota_k$ and ι_k is the Clifford dual unit.

The dual numbers are not often used and we are briefly reminded of their algebraic properties. Each of the dual units is not equal to zero, $\iota_k \neq 0$; a different dual unit obeys the commutative law of multiplication $\iota_k \iota_m = \iota_m \iota_k \neq 0, k \neq m$; the square of a dual unit is always equal to zero, $\iota_k^2 = 0$. Division of a real or complex number by the dual unit is not defined, but the division of a

Table I. The nine Cayley–Klein plane geometries.

	$j_1 = 1$	$j_1 = \iota_1$	$j_1 = i$
$j_1 = 1$	Elliptic	Euclidean	Lobachevskian (hyperbolic)
$j_2 = \iota_2$	Semielliptic (co-Euclidean)	Galilean	Semihyperbolic (co-Minkowskian)
$j_2 = i$	Anti-de Sitter	Minkowskian	De Sitter

dual unit by itself is equal to the real unit $\iota_k/\iota_k = 1$. The last property is equivalent to the fact that the equation $a\iota_k = b\iota_k$ has only one solution $a = b$ in the real or complex number field. A function of a dual argument is defined by its Taylor expansion.

The quadratic form of the Euclidean space $\mathbf{R}_{n+1}$ is transformed under the map (1) into the following quadratic form of the Cayley-Klein space $\mathbf{R}_{n+1}(\mathbf{j})$:

$$(\mathbf{x}, \mathbf{x}) = x_0^2 + \sum_{k=1}^n x_k^2 \prod_{m=1}^k j_m^2, \quad (2)$$

where $\mathbf{x} = (x_0, x_1, \dots, x_n)$. The set of parameters $\mathbf{j}$ gives all 3^n $(n+1)$ -dimensional Cayley-Klein spaces $\mathbf{R}_{n+1}(\mathbf{j})$ or all 3^n Cayley-Klein geometries (spaces of the constant curvature), which are realized on the spheres $S_n(\mathbf{j}) = \{\mathbf{x} \in \mathbf{R}_{n+1}(\mathbf{j}) | (\mathbf{x}, \mathbf{x}) = 1\}$. Table I gives the identifications of the nine Cayley-Klein plane geometries with the values of the parameters $\mathbf{j}$.

It must be emphasized that usually the spaces with the identical signature are not distinguished, i.e., the space $\mathbf{R}_3(1, i)$ with the form $x_0^2 + x_1^2 - x_2^2$ and the space $\mathbf{R}_3(i, 1)$ with the form $x_0^2 - x_1^2 - x_2^2$ are regarded as the same space. We have fixed the numbers of coordinate axes and for us $\mathbf{R}_3(1, i)$ and $\mathbf{R}_3(i, 1)$ are the different spaces.

The orthogonal Cayley-Klein group $SO_{n+1}(\mathbf{j})$ is defined [11],[13] as the group of all keeping invariant the form (2) transformations in the space $\mathbf{R}_{n+1}(\mathbf{j})$. The map (1) induces the transformations of the group SO_{n+1} into the group $SO_{n+1}(\mathbf{j})$ and, respectively, transformation of the algebra so_{n+1} into the algebra $so_{n+1}(\mathbf{j})$. The generators X^* of so_{n+1} are transformed as follows:

$$X_{rs}(\mathbf{j}) = \left(\prod_{m=r+1}^s j_m \right) X_{rs}^*(\rightarrow), \quad r < s, \quad (3)$$

where $X^*(\rightarrow)$ denotes the Wigner-Inonu [2] singular transformed generators and the products $\prod j_m$ play the role of zero tending parameters, when some parameters j_m are equal to the dual units. When some parameters j_m are equal to the imaginary unit, then Eqs. (3) give the transformations, corresponding to the Weyl unitary trick [7] or the analytic continuations of the representations. Therefore, two different operations in group theory are naturally unified and uniformly treated by Eqs. (3).

If we look at the group elements $g^* = \exp(a^* X^*)$ instead of the generators X^* , we observe that the transformations (3) correspond to replacement of the real group parameters a^* by the group parameters $a - a^* \prod j_m$, which are dual ones in the case of contractions and are imaginary ones in the case of analytic continuations. In other words, the Weyl trick and the Wigner-Inonu contraction have the same nature, namely the continuation of the real group parameters into the complex or dual number field.

If X_{rs}^* , $r, s = 0, 1, \dots, n$ are the rotation generators in the two-dimensional coordinate planes $\{x_r, x_s\}$, then the commutation relations of the classical orthogonal algebra so_{n+1} are as follows:

$$[X_{kr}^*, X_{pq}^*] = \delta_{rp} X_{kq}^* + \delta_{kq} X_{rp}^* - \delta_{rq} X_{kp}^* - \delta_{kp} X_{rq}^*. \quad (4)$$

The generators X_{rs}^* are transformed under the map (1) according to Eqs. (3) and we obtain from Eqs. (4) the following commutation relations of the orthogonal Cayley-Klein algebra $so_{n+1}(\mathbf{j})$:

$$[X_{r_1 s_1}, X_{r_2 s_2}] = \begin{cases} \left(\prod_{l=r_1+1}^{s_1} j_l \right) X_{s_1 s_2}, & r_1 = r_2, & s_1 < s_2, \\ \left(\prod_{l=r_2+1}^{s_2} j_l \right) X_{r_1 r_2}, & r_1 < r_2, & s_1 = s_2, \\ -X_{r_1 s_2}, & r_1 < r_2 = s_1 < s_2. \end{cases} \quad (5)$$

We write out only nonzero commutators.

The map (1) also induces the corresponding transformations of the invariant (or Casimir) operators of so_{n+1} . As is well known, there are $[n+1]/2$ independent Casimir operators of the algebra so_{n+1} , where $[x]$ is the integral part of the number x . According to the Gel'fand [16] definition the Casimir operator C_{2p}^* is the sum of all $2p$ -dimensional principal minors of the antisymmetric matrix A , which is constructed from the generators X_{rs}^* , namely $(A)_{rs} = X_{rs}^*$, $(A)_{sr} = -X_{rs}^*$. For even $n = 2k$, all the Casimir operators are obtained in this manner for $p = 1, 2, \dots, k$. The square root of the determinant of the matrix A , namely

$$C_{k+1}'^*(X_{rs}^*) = \sum_{l_1, \dots, l_{n+1}}^n \epsilon_{l_1, \dots, l_{n+1}} X_{l_1 l_2}^* X_{l_3 l_4}^* \cdots X_{l_n l_{n+1}}^*, \quad (6)$$

where $\epsilon_{l_1, \dots, l_{n+1}}$ is the antisymmetric unit tensor, is added to the previous Casimir operators in the case of odd $n = 2k + 1$.

As was explained in detail earlier, [17] the Casimir operators of the algebra $so_{n+1}(\mathbf{j})$ are obtained from those of so_{n+1} if we replace the generators X^* by the generators X according to (3) and multiply the obtained expressions by the least common denominators of the coefficients of the addends in the sum. These least common denominators are easily found with the help of mathematical induction. Then the transformation laws of Casimir operators are as follows:

$$C_{2p}(\mathbf{j}; X_{rs}) = C_{2p}^* \left(x_{rs}^* \rightarrow X_{rs} \prod_{l=r+1}^s j_l^{-1} \right) \times \prod_{m=1}^{p-1} j_m^{2m} j_{n-m+1}^{2m} \prod_{l=p}^{n-p+1} j_l^{2p}, \quad (7)$$

$$C_{k+1}'(\mathbf{j}; X_{rs}) = C_{k+1}'^* \left(x_{rs}^* \rightarrow X_{rs} \prod_{l=r+1}^s j_l^{-1} \right) j_{(n+1)/2}^{(n+1)/2} \times \prod_{m=1}^{(n-1)/2} j_m^{2m} j_{n-m+1}^{2m}. \quad (8)$$

It will be noted that the structure of the transformations (3) is the same for all representations of the orthogonal algebra. Only the generators $X^*(\rightarrow)$ are denned in a different way for a different representation [17]. In the case of the Gel'fand-Tsetlin representations the singular transformed generators $X^*(\rightarrow)$ are specified by the transformations of the components of the patterns.

3 Representations of the algebra $so_{n+1}(\mathbf{j})$ in the Gel'fand-Tsetlin basis

In contrast with the standard Gel'fand-Tsetlin notation1 we regard the basis vectors of an arbitrary irreducible representation of $so_{n+1}(\mathbf{j})$, which are completely characterized by the chain of subalgebras $so_{n+1}(\mathbf{j}) \supset so_n(j_2, \dots, j_n) \supset \dots \supset so_3(j_{n-1}, j_n) \supset so_2(j_n)$, where $so_{n+1}(\mathbf{j}) = \{x_{sp}, s < p, s, p = 0, 1, \dots, n\}$, $so_n(j_2, \dots, j_n) = \{X_{sp}, s < p, s, p = 1, 2, \dots, n\}, \dots, so_3(j_{n-1}, j_n) = \{X_{sp}, s < p, s, p = n-2, \dots, n\}, so_2 = \{X_{n-1, n}\}$. The standard notation is obtained from ours, if we change the indexes s and p of the generators by $n+1-s$ and $n+1-p$, respectively.

From the commutation relations (5) it follows that the irreducible representation is completely denned by the generators $X_{p-1, p}$, $p = 1, 2, \dots, n$. These generators are transformed under the map (1) as follows:

$$X_{p-1, p} = j_p X_{p-1, p}^*(\rightarrow). \quad (9)$$

The singular transformed generators $X^*(\rightarrow)$ are specified by the transformations of the components of the Gel'fand-Tsetlin patterns and the main problem now is to define these transformation laws.

For the algebra so_{2k+2} , i.e., $n = 2k + 1$ the pattern is

$$|m^*\rangle = \left| \begin{array}{cccccc} m_{1,2k+1}^* & & m_{2,2k+1}^* & \cdots & m_{k,2k+1}^* & & m_{k+1,2k+1}^* \\ & m_{1,2k}^* & \cdots & m_{2,2k}^* & \cdots & m_{k,2k}^* & \\ & m_{1,2k-1}^* & \cdots & m_{2,2k-1}^* & \cdots & m_{k,2k-1}^* & \\ & & m_{1,2k-2}^* & \cdots & m_{k-1,2k-2}^* & & \\ & & m_{1,2k-3}^* & \cdots & m_{k-1,2k-3}^* & & \\ & & \cdots & \cdots & \cdots & & \\ & & m_{14}^* & & m_{24}^* & & \\ & & m_{13}^* & & m_{23}^* & & \\ & & & m_{12}^* & & & \\ & & & m_{11}^* & & & \end{array} \right\rangle$$

$$m_{1,2k+1}^* \geq m_{1,2k+1}^* \geq m_{2,2k+1}^* \geq m_{2,2k}^* \cdots \geq m_{k,2k+1}^* m_{k,2k}^* \geq |m_{k+1,2k+1}^*|$$

$$m_{1,2k}^* \geq m_{1,2k-1}^* \geq m_{2,2k}^* \geq m_{2,2k-1}^* \geq \cdots \geq m_{k,2k-1}^* \geq -m_{k,2k}^*,$$

$$m_{1,2k-1}^* \geq m_{1,2k-2}^* \geq m_{2,2k-1}^* \geq \cdots \geq m_{k-1,2k-2}^* \geq |m_{k,2k-1}^*|$$

$$m_{i,2p+1}^* \geq m_{i,2p}^* \geq m_{i+1,2p+1}^*,$$

$$m_{p,2p+1}^* \geq m_{p,2p}^* \geq |m_{p+1,2p+1}^*|,$$

$$m_{p,2p}^* \geq m_{i,2p-1}^* \geq m_{i+1,2p}^*,$$

$$m_{p,2p}^* \geq m_{p,2p-1}^* \geq -m_{p,2p}^*,$$

$$i = 1, 2, \dots, p-1, p = 1, 2, \dots, k, \quad (10)$$

and for the algebra so_{2k+1} i.e., $n = 2k$ the pattern is

$$|m^*\rangle = \left| \begin{array}{cccc} m_{1,2k}^* & & m_{2,2k}^* & \cdots & m_{k,2k}^* \\ m_{1,2k-1}^* & & m_{2,2k-1}^* & \cdots & m_{k,2k-1}^* \\ & m_{1,2k-2}^* & \cdots & m_{k-1,2k-2}^* & \\ & m_{1,2k-3}^* & \cdots & m_{k-1,2k-3}^* & \\ & & \cdots & & \\ & m_{14}^* & & m_{24}^* & \\ & m_{13}^* & & m_{23}^* & \\ & & m_{12}^* & & \\ & & m_{11}^* & & \end{array} \right\rangle$$

$$m_{1,2k}^* \geq m_{1,2k-1}^* \geq m_{2,2k}^* \geq m_{2,2k-1}^* \cdots \geq m_{k,2k-1}^* \geq -m_{k,2k}^*,$$

$$m_{1,2k-1}^* \geq m_{1,2k-2}^* \geq m_{2,2k-1}^* \geq \cdots \geq m_{k-1,2k-2}^* \geq |m_{k,2k-1}^*|, \quad (11)$$

and so on. The components m^* in the top row of the patterns (10)–(11) characterize the irreducible representations of so_{n+1} . The components in the next row characterize one of the possible irreducible representations of so_n that is contained in the specific irreducible representation of so_{n+1} , and so forth for successive subalgebras of the chain. Thus the components in each row characterize one of the possible irreducible representations of a

specific subalgebra. The eigenvalues of the invariant (or Casimir) operators of so_{n+1} have been obtained by Perelomov, Popov [18] and Leznov, Malkin, and Man'ko [19] and are given by

$$C_q^*(m^*) = Tr\{(a^*)^q E\}, \quad (12)$$

where E is $n \times n$ matrix with each matrix element equal to 1. For the algebra so_{2k+2} the matrix a^* is in the form

$$a_{ps}^* = [m_{p,2k+1}^* + (k+1)\epsilon_p - p + k]\delta_{ps} - \theta_{sp} + \frac{1}{2}(1 + \epsilon_p)\delta_{p,-s}, \quad (13)$$

$$p, s = 1, 2, \dots, k+1, -(k+1), \dots, -2, -1,$$

and for the algebra sp_{2k+1} in the form

$$a_{ps}^* = [m_{p,2k}^* + (k + \frac{1}{2})\epsilon_p - p + k - \frac{1}{2}]\delta_{ps} - \theta_{sp} + \frac{1}{2}(1 + \epsilon_p)\delta_{p,-s}, \quad (14)$$

$$p, s = 1, 2, \dots, k, 0, -k, \dots, -2, -1,$$

where $\theta_{sp} = 1$ if $s < p$ and $\theta_{sp} = 0$ if $s \geq p$; $\epsilon_p = 0$ if $p = 0$, $\epsilon_p = 1$ if $p > 0$, and $\epsilon_p = 1$ if $p < 0$. In particular, the eigenvalues of the second-order Casimir operators are as follows:

$$X_2^* - 2 \sum_p m_p^* [m_p^* + 2(k - p + q)], \quad (15)$$

where $m_p^* = m_{p,2k+1}^*$, $q = 1$ for $so_{2(k+1)}$ and $m_p^* = m_{p,2k}^*$, $q = \frac{1}{2}$ for so_{2k+1} .

The irreducible representation of so_{n+1} and the eigenvalues of the Casimir operators are completely characterized by the components in the top row of the patterns (10)–(11). For the nondegenerate irreducible representation the eigenvalues of all Casimir operators differ from zero. If we want to obtain under all contractions (including the maximal dimension contraction, when all parameters j_o are equal to dual units) the general nondegenerate representations of the contracted algebras, we must choose the laws of transformations of the components of the Gel'fand-Tsetlin patterns in such a way that the eigenvalues of all Casimir operators differ from zero for all contractions. The eigenvalues of the Casimir operators are transformed under the map ψ as the operators themselves. Therefore from Eqs. (7) and (15) we conclude that the component $m_{1,2k+1}^*$, which defines the eigenvalues of the second-order Casimir operator, is transformed as follows:

$$m_{1,2k+1} = m_{1,2k+1}^* \prod_{l=1}^{2k+1} j_l \equiv m_{1,2k+1}^* J_{1,2k+1}. \quad (16)$$

The transformation law of the component $m_{p,2k+1}^*$ coincides with the transformation law of the algebraic construction $\{C_{2p}^*/C_{2(p-1)}^*\}^{1/2}$ and is equal to

$$m_{p,2k+1} = m_{p,2k+1}^* \prod_{l=p}^{2(k+1)-p} j_l \equiv m_{p,2k+1}^* J_{p,2k+1}, \quad p = 2, 3, \dots, k. \quad (17)$$

The last component $m_{k+1,2k+1}^*$ is transformed as the ratio $C_{k+1}^*/\{C_{2k}^*\}^{1/2}$, i.e., $m_{p,2k+1} = m_{p,2k+1}^* j_{k+1}$. This formula is obtained from Eqs. (17) for

$p = k + 1$ and Eqs. (16) are also obtained from (17) for $p = 1$, therefore all components of the top row of the pattern (10) are transformed according to Eqs. (17) with $p = 1, 2, \dots, k + 1$.

The transformation laws of the components of the next row of the pattern (10) are found quite analogous to the previous case with the help of the Casimir operators of the subalgebra $so_{2k+1}(j_2, \dots, j_{2k+1})$ and are given by Eqs. (17), except the product begins from $p + 1$. Then the transformation laws of the components of the Gel'fand-Tsetlin patterns are as follows:

$$\begin{aligned} m_{p,2s+1} &= m_{p,2s+1}^* \prod_{l=p+2(k-s)}^{2(k+1)-p} j_l \equiv m_{p,2s+1}^* J_{p,2s+1}, \quad s = 0, 1, \dots, k, \quad p = 1, 2, \dots, s+1, \\ m_{p,2s} &= m_{p,2s}^* \prod_{l=p+2(k-s)+1}^{2(k+1)-p} j_l \equiv m_{p,2s}^* J_{p,2s}, \quad s = 1, 2, \dots, k, \quad p = 1, 2, \dots, s, \end{aligned} \quad (18)$$

for the algebra $so_{2k+2}(\mathbf{j})$ and

$$\begin{aligned} m_{p,2s} &= m_{p,2s}^* \prod_{l=p+2(k-s)}^{2k+1-p} j_l \equiv m_{p,2s}^* m_{p,2s}^* J_{p,2s}, \quad s = 1, 2, \dots, k, \quad p = 1, 2, \dots, s, \\ m_{p,2s-1} &= m_{p,2s-1}^* \prod_{l=p+2(k-s)+1}^{2k+1-p} j_l \equiv m_{p,2s-1}^* J_{p,2s-1}, \end{aligned} \quad (19)$$

for the algebra $so_{2k+1}(\mathbf{j})$ or in a unified manner,

$$m_{pr} = m_{pr}^* \prod_{l=n+p-r}^{n+1-p} j_l \equiv m_{pr}^* J_{pr}, \quad p < r, \quad (20)$$

for the algebra $so_{n+1}(\mathbf{j})$.

Equations (10) and (18) give the patterns $|m\rangle$ of type (10), but the transformed components m_{ps} may be equal, in general, to complex or dual values. Using Eqs. (9) and (18) we obtain from the well-known [20] explicit generators of so_{n+1} the generators of the irreducible representations of $so_{n+1}(\mathbf{j})$ in the form

$$\begin{aligned} X_{2(k-s)+1,2(k-s+1)} |m\rangle &= \sum_{p=1}^s J_{p,2s-1}^{-1} \{A(m_{p,2s-1}) |m_{p,2s-1} + J_{p,2s-1}\rangle - \\ &\quad - A(m_{p,2s-1}) - J_{p,2s-1} |m_{p,2s-1} - J_{p,2s-1}\rangle\}, \quad s = 1, 2, \dots, k, \\ X_{2(k-s)+2(k-s)+1} |m\rangle &= iD_{2s} |m\rangle + \sum_{p=1}^s J_{p,2s}^{-1} \{B(m_{p,2s}) |m_{p,2s} + J_{p,2s}\rangle - \\ &\quad - B(m_{p,2s}) - J_{p,2s} |m_{p,2s} - J_{p,2s}\rangle\}, \quad s = 0, 1, \dots, k-1, \end{aligned} \quad (21)$$

where

$$\begin{aligned} D_{2s} &= \frac{\prod_{p=1}^s l_{p,2s-1} \prod_{p=1}^{s+1} l_{p,2s+1}}{\prod_{p=1}^s l_{p,2s} (l_{p,2s} - J_{p,2s})}, \\ B(m_{p,2s}) &= \left\{ \frac{\prod_{r=1}^{p-1} (l_{r,2s-1}^2 - l_{p,2s}^2 a_{r,p,s}^2) \prod_{r=p}^s (l_{r,2s-1}^2 a_{r,p,s}^{-2} - l_{p,2s}^2)}{l_{p,2s}^2 (4l_{p,2s}^2 - J_{p,2s}^2) \prod_{r=1}^{p-1} (l_{r,2s}^2 - l_{p,2s}^2 c_{r,p,s}^2) [(l_{r,2s} - J_{r,2s})^2 - l_{p,2s}^2 c_{r,p,s}^2]} \right\} \end{aligned}$$

$$\begin{aligned}
& \times \frac{\prod_{r=1}^p (l_{r,2s+1}^2 - l_{p,2s}^2 b_{r,p,s}^2) \prod_{r=p+1}^{s+1} (l_{r,2s+1}^2 b_{r,p,s}^{-2} - l_{p,2s}^2)}{\prod_{r=p+1}^s (l_{r,2s}^2 c_{r,p,s}^{-2} - l_{p,2s}^2) [(l_{r,2s} - J_{r,2s})^2 c_{r,p,s}^{-2} - l_{p,2s}^2]} \Bigg\}^{1/2}, \\
A(m_{p,2s-1}) &= \frac{1}{2} \left\{ \frac{\prod_{r=1}^{p-1} (l_{r,2s-2} - l_{p,2s-1} a_{r,p,s-1/2} - J_{2,2s-1}) (l_{r,2s-2} + l_{p,2s-1} a_{r,p,s-1/2})}{\prod_{r=p}^s (l_{r,2s-1}^2 c_{r,p,s-1/2}^{-2} - l_{p,2s-1}^2)} \right. \\
& \times \frac{\prod_{r=p}^{s-1} (l_{r,2s-2} a_{r,p,s-1/2}^{-1} - l_{p,2s-1} - J_{p,2s-1}) (l_{r,2s-2} a_{r,p,s-1/2}^{-1} + l_{p,2s-1})}{[l_{r,2s-1}^2 - (l_{p,2s-1} + J_{p,2s-1}) c_{r,p,s-1/2}^2]} \\
& \frac{\prod_{r=1}^p (l_{r,2s} - l_{p,2s-1} b_{r,p,s-1/2} - J_{r,2s}) (l_{r,2s-1} b_{r,p,s-1/2})}{\prod_{r=p+1}^s (l_{r,2s-1}^2 c_{r,p,s-1/2}^{-2} - l_{p,2s-1}^2)} \\
& \left. \times \frac{\prod_{r=p+1}^s (l_{r,2s} b_{r,p,s-1/2}^{-1} - l_{p,2s-1} - J_{p,2s-1}) (l_{r,2s} b_{r,p,s-1/2}^{-1} + l_{p,2s-1})}{[l_{r,2s-1}^2 c_{r,p,s-1/2}^{-2} - (l_{p,2s-1} + J_{p,2s-1})^2]} \right\}^{1/2}, \\
a_{r,p,s} &= \frac{J_{r,2s-1}}{J_{p,2s}}, \quad b_{r,p,s} = \frac{J_{r,2s+1}}{J_{p,2s}}, \quad c_{r,p,s} = \frac{J_{r,2s}}{J_{p,2s}} \\
l_{p,2s} &= m_{p,2s} + (s - p + 1) J_{p,2s}. \tag{22}
\end{aligned}$$

Since the generators (21) have been built from the generators of so_{n_1} with the help of the transformations (3) they satisfy the commutation relations of $so_{n+1}(\mathbf{j})$. Although the initial operators of the irreducible representations of so_{n+1} are Hermitian, the operators (21), generally speaking, are not Hermitian. We must impose additionally the Hermitic requirements on the operators (21), namely $X_{rs}^+ = -X_{rs}$, $r < s$, $r, s = 0, 1, \dots, n$ if we want to obtain the Hermitic representation. The corresponding requirements for the matrix elements are in the form

$$\langle n | X_{rs} | m \rangle = -\overline{\langle m | X_{rs} | n \rangle}, \tag{23}$$

where the overbar denotes the complex conjugation.

4 Contractions and analytic continuation of the representations of $so_3(j_1, j_2)$

Under the map (1) the generators of the rotation algebra so_3 are transformed according to (2) as follows:

$$X_{01} = j_1 X_{01}^*(\rightarrow), \quad X_{02} = j_1 j_2 X_{02}^*(\rightarrow), \quad X_{12} = j_2 X_{12}^*(\rightarrow), \tag{24}$$

and the resulting generators have the commutation relations,

$$[X_{12}, X_{01}] = X_{02}, \quad [X_{02}, X_{12}] = j_2^2 X_{01}, \quad [X_{01}, X_{02}] = j_1^2 X_{12}, \tag{25}$$

of the Cayley–Klein algebra $so_3(j_1, j_2)$. The corresponding Cayley–Klein geometries are in Table I. The only Casimir operator of the algebra so_3 is given by

$$C_2^* = X_{01}^{*2} + X_{02}^{*2} + X_{12}^{*2}, \tag{26}$$

and is transformed according to Eqs (7) to the Casimir operator of $so_3(\mathbf{j})$, $\mathbf{j} = (j_1, j_2)$ in the following manner:

$$C_2(\mathbf{j}) = j_1^2 j_2^2 C_2^*(\rightarrow) = j_1^2 j_2^2 (j_1^{-2} X_{01}^2 + j_1^{-2} j_2^{-2} X_{02}^2 + j_2^{-2} X_{12}^2) =$$

$$= j_2^2 X_{01}^2 + X_{02}^2 + j_1^2 X_{12}^2. \quad (27)$$

The Casimir operator of the two-dimensional Galilean algebra is well known $C_2(\iota_1, \iota_2) = P^2$; for the two-dimensional Poincare algebra it is given by the expression $C_2(\iota_1, i) = P^2 - H^2$ and the Casimir operator of the pseudoorthogonal algebra $so(2, 1)$ is equal to $C_2(i, 1)H^2 + P^2 - K^2$. All these Casimir operators are given by Eq. (27) under the appropriate values of parameters $(\mathbf{j})$ and identifications of the generators as $H \equiv X_{01}$, $P \equiv X_{02}$, $K \equiv X_{12}$.

The components of the pattern

$$|m^*\rangle = \begin{vmatrix} m_{12}^* \\ m_{11}^* \end{vmatrix}$$

are transformed according to Eqs.(10) as follows:

$$m_{12} = j_1 j_2 m_{12}^*, \quad m_{11} = j_2 m_{11}^*, \quad (28)$$

and Eqs.(21) for $n = 2$ together with commutation relations (25) give the following generators of the irreducible representations of the algebra $so_3(j_1, j_2)$:

$$\begin{aligned} X_{01}|m\rangle &= (1/2j_2)(\alpha^+|m_{11} + j_2\rangle - \alpha^-|m_{11} - j_2\rangle), \\ X_{02}|m\rangle &= (i/2j_2)(\alpha^+|m_{11} + j_2\rangle + \alpha^-|m_{11} - j_2\rangle), \\ X_{12}|m\rangle &= im_{11}|m\rangle, \\ \alpha^\pm &= \sqrt{(m_{12} \mp j_1 m_{11})(m_{12} \pm j_1 m_{11} + j_1 j_2)}. \end{aligned} \quad (29)$$

It is well known that the eigenvalues of the Casimir operator C_2^* on the irreducible Gel'fand–Tsetlin representation of so_3 are given by $C_2^* = m_{12}^*(m_{12}^* + 1)$. Then the eigenvalues of the Casimir operator C_2 of the representation (29) of $so_3(j_1, j_2)$ are easily obtained with the help of Eqs.(27) and (28) in the form $C_2(\mathbf{j}) = m_{12}(m_{12} + j_1 j_2)$.

The $j_1 = \iota_1$ contracted algebra $so_3(\iota_1, j_2)$ has the structure of the semidirect sum

$$so_3(\iota_1, j_2) \equiv iso_2(j_2) = \{X_{01}, X_{02}\}(\oplus \{X_{12}\}), \quad (30)$$

and is the inhomogeneous (or Euclidean) orthogonal algebra iso_2 for $j_2 = 1$. We easily obtain from Eqs. (29) the generators of the representation of $so_3(\iota_1, j_2)$ in the form

$$\begin{aligned} X_{01}|m\rangle &= (m_{11}/2j_2)(|m_{11} + j_2\rangle - |m_{11} - j_2\rangle), \\ X_{12}|m\rangle &= im_{11}|m\rangle, \\ X_{02}|m\rangle &= (i/2)m_{11}(|m_{11} + j_2\rangle + |m_{11} - j_2\rangle). \end{aligned} \quad (31)$$

These generators are Hermitian in the case of $j_2 = 1$ if the component m_{11} are integers or half-integers and the component m_{12} is non-negative real $m_{12} \geq 0$. The eigenvalues of the Casimir operator are $C_2(\iota_1, j_2) = m_{12}^2$.

The $j_2 = \iota_2$ contracted algebra $so_3(j_1, \iota_2)$ has the structure of the semidirect sum

$$so_3(j_1, \iota_2) \equiv iso_2(j_1) = \{X_{02}, X_{12}\}(\oplus \{X_{01}\}), \quad (32)$$

and is isomorphic with the inhomogeneous orthogonal algebra iso_2 for $j_1 = 1$. The generators of the representation of $so_3(j_1, \iota_2)$ are obtained from Eqs. (29) in the form

$$\begin{aligned} X_{12}|m\rangle &= im_{11}|m\rangle, \quad X_{02}|m\rangle = i\sqrt{m_{12}^2 - j_1^2 m_{11}^2}|m\rangle, \\ X_{01}|m\rangle &= \sqrt{m_{12}^2 - j_1^2 m_{11}^2} \frac{\partial|m\rangle}{\partial m_{11}} - j_1^2 \frac{m_{11}}{2\sqrt{m_{12}^2 - j_1^2 m_{11}^2}}|m\rangle, \end{aligned} \quad (33)$$

where $m_{11}, m_{12} \in \mathbf{R}$, $m_{12} \geq 0$ and $|m_{11}| \leq m_{12}$ for $j_1 = 1$. The last inequality follows from the Hermitic requirements (23). The eigenvalues of the Casimir operator are $C_2(j_1, \iota_2) = m_{12}^2$ and coincide with $C_2(\iota_2, j_2) = m_{12}^2$.

Let us now observe that the algebras $so_3(\iota_1, 1)$ and $so_3(1, \iota_2)$ are isomorphic to each other and both isomorphic to the inhomogeneous algebra iso_2 . Then the operators (31) realize the irreducible representations of iso_2 in the discrete basis

$$\left| \begin{array}{c} m_{12} \\ m_{11} \end{array} \right\rangle \equiv \left| \begin{array}{c} k \\ m_{11} \end{array} \right\rangle,$$

$k \geq 0, m_{11}$ integers, $-\infty < m_{11} < \infty$ which is characterized by the chain of subalgebras $iso_2 \equiv so_3(\iota_1, 1) \supset so_2(1)$, where $so_2(1) = \{X_{12}\}$ is the compact one-dimensional subalgebra, while the operators (33) realize the irreducible representations of iso_2 in the continuous basis

$$\left| \begin{array}{c} m_{12} \\ m_{11} \end{array} \right\rangle \equiv \left| \begin{array}{c} k \\ p \end{array} \right\rangle,$$

$k \geq 0, |p| \leq k$, which is characterized by the chain $iso_2 \equiv so_3(1, \iota_2) \supset so_2(\iota_2)$, where $so_2(i) = \{X_{12}\}$ is now the contracted noncompact one-dimensional algebra. Thus our approach gives the irreducible representations of the same contracted algebras in a different basis.

The two-dimensional contraction of the $so_3(\mathbf{j})$ may be obtained by three different ways. We may put $j_1 = \iota_1, j_2 = \iota_2$ in Eqs. (29), or we may put $j_2 = \iota_2$ in Eqs. (31), or we may put $j_1 = \iota_1$ in Eqs.(33), but we always get the same result, namely the representations of the Galilean algebra $so(\iota_1, \iota_2)$,

$$X_{01}|m\rangle = m_{12} \frac{\partial|m\rangle}{\partial m_{11}}, \quad X_{02}|m\rangle = im_{12}|m\rangle, \quad X_{12}|m\rangle = im_{11}|m\rangle, \quad (34)$$

where the components m_{11}, m_{12} are real, $m_{12} \geq 0, -\infty < m_{11} < \infty$ in the case of the Hermitic representation. The eigenvalues of the Casimir operator are $C_2(l_1, l_2) = m_{12}^2$ and coincide with those of $C_2(l_1, j_2) = C_2(j_1, l_2) = m_{12}^2$. The result of the application of the generator to the derivate $\partial|m\rangle$ *partial* m_{11} is obtained by application of the generator to both sides of the equation

$$\frac{\partial|m\rangle}{\partial m_{11}} = \frac{1}{2\iota_2}(|m_{11} + \iota_2\rangle - |m_{11} - \iota_2\rangle),$$

in particular,

$$X_{12} \frac{\partial|m\rangle}{\partial m_{11}} = im_{11} \frac{\partial|m\rangle}{\partial m_{11}} + i|m\rangle.$$

Let us introduce instead of (29) the new generators $J_{\pm}, J_3$ of $so_3(\mathbf{j})$,

$$J_{\pm}|m\rangle = (\pm j_2 X_{01} - i X_{02})|m\rangle = \alpha^{\pm}|m_{11} \pm j_2\rangle, \quad J_3|m\rangle = -i X_{12}|m\rangle = m_{11}|m\rangle. \quad (35)$$

The infinite-dimensional representation of the analytic continued algebra $so_3(i, j_2) \equiv so(1, 2)$ is described by Eqs. (??) with $j_1 = i$ and $m_{12} = a + ib \in \mathbf{C}$. The Hermitic requirements (23) for generators $J_{\pm}$ are in the form $(J_{-})^{+}$ and give the following irreducible representations of the pseudoorthogonal algebra $so_3(i, j_2) \equiv so(1, 2)$:

(i) $l = a - i/2$, $m \in \mathbf{Z}$, the principal series,

$$J_{\pm}|l, m\rangle = \sqrt{(1/2 \pm m + ia)(1/2 \pm m - ia)}|l, m \pm 1\rangle; \quad (36)$$

(ii) $l = ib$, $-1 < b < 0$, $m \in \mathbf{Z}$, the supplementary series,

$$J_{\pm}|l, m\rangle = i\sqrt{(b \pm m)(b \pm m + 1)}|l, m \pm 1\rangle; \quad (37)$$

(iii) $l = ib$, $b \in \mathbf{N}$, $\mathbf{N} + \frac{1}{2}$, $m \in \{b, b + 1, \dots\}$ or $m \in \{-b, -b - 1, \dots\}$, the discrete series with the generators $J_{\pm}$ as in (36).

Contractions of the representations of the algebra $so_4(\mathbf{j})$ For the algebra $so(4; \mathbf{j})$, $\mathbf{j} = (j_1, j_2, j_3)$ the transformation laws of the generators and of the components of the patterns are defined by Eqs. (2) and (18) in the following manner:

$$\begin{aligned} X_{01} &= j_1 X_{01}^*(\rightarrow), & X_{02} &= j_1 j_2 X_{02}^*(\rightarrow), & X_{03} &= j_1 j_2 j_3 X_{03}^*(\rightarrow), \\ X_{12} &= j_2 X_{12}^*(\rightarrow), & X_{13} &= j_2 j_3 X_{13}^*(\rightarrow), & X_{23} &= j_3 X_{23}^*(\rightarrow), \end{aligned} \quad (38)$$

$$|m\rangle = \begin{vmatrix} m_{13} & m_{23} \\ & m_{12} \\ & & m_{11} \end{vmatrix} \equiv \begin{vmatrix} m_{12} \\ m_{11} \end{vmatrix},$$

$$m_{13} = j_1 j_2 j_3 m_{13}^*, \quad m_{23} = j_2 m_{23}^*, \quad m_{12} = j_2 j_3 m_{12}^*, \quad m_{11} = j_3 m_{11}^*$$

and the commutation relations of $so_4(\mathbf{j})$ according to eqs. (5) are

$$\begin{aligned} [X_{01}, X_{02}] &= j_1^2 X_{12}, & [X_{01}, X_{03}] &= j_1^2 X_{13}, & [X_{02}, X_{03}] &= j_1^2 j_2^2 X_{23}, \\ [X_{01}, X_{12}] &= -X_{02}, & [X_{01}, X_{13}] &= -X_{03}, & [X_{02}, X_{12}] &= j_2^2 X_{01}, \\ [X_{03}, X_{13}] &= j_2^2 j_3^2 X_{01}, & [X_{03}, X_{23}] &= j_3^2 X_{02}, & [X_{02}, X_{23}] &= -X_{03}, \\ [X_{12}, X_{13}] &= j_2^2 X_{23}, & [X_{12}, X_{23}] &= -X_{13}, & [X_{13}, X_{23}] &= j_3^2 X_{12}, \end{aligned} \quad (39)$$

Two Casimir operators of so_4 are transformed to the Casimir operators of the Cayley–Klein algebra $so(4; \mathbf{j})$,

$$C_2(\mathbf{j}) = j_1^2 j_2^2 j_3^2 C_2^*(\rightarrow) = j_2^2 j_3^2 X_{01}^2 + j_3^2 X_{02}^2 + X_{03}^2 + j_1^2 j_3^2 X_{12}^2 + j_1^2 X_{13}^2 + j_1^2 j_2^2 X_{23}^2, \quad (40)$$

$$C_2'(\mathbf{j}) = j_1 j_2^2 j_3 C_2'^*(\rightarrow) = j_2^2 X_{01} X_{23} + X_{03} X_{12} - X_{02} X_{13},$$

and have the following eigenvalues:

$$C_2(\mathbf{j}) = m_{13}(m_{13} + 2j_1 j_2 j_3) + j_1^2 j_3^2 m_{23}^2, \quad C_2'(\mathbf{j}) = -(m_{13} + 2j_1 j_2 j_3)m_{23}. \quad (41)$$

The generators of $so(4; \mathbf{j})$ are given by Eqs.(21)-(22) in the form

$$\begin{aligned} X_{23}|m\rangle &= im_{11}|m\rangle, \quad a(m_{11}) = \sqrt{(m_{12} - j_2 m_{11} + j_2 j_3)(m_{12} + j_2 m_{11})}, \\ X_{12}|m\rangle &= \frac{1}{2j_3} a(m_{11} + j_3) \begin{vmatrix} m_{12} \\ m_{11} + j_3 \end{vmatrix} - \frac{1}{2j_3} a(m_{11}) \begin{vmatrix} m_{12} \\ m_{11} - j_3 \end{vmatrix}, \end{aligned}$$

$$\begin{aligned}
X_{01}|m\rangle &= m_{11}A(m_{12}|m\rangle - \frac{1}{j_2j_3}D(m_{12})\sqrt{m_{12}^2 - j_2^2m_{11}^2} \left| \begin{matrix} m_{12} - j_2j_3 \\ m_{11} \end{matrix} \right\rangle + \\
&+ \frac{1}{j_2j_3}D(m_{12} + j_2j_3)\sqrt{m_{12} + j_2j_3)^2 - j_2^2m_{11}^2} \left| \begin{matrix} m_{12} + j_2j_3 \\ m_{11} \end{matrix} \right\rangle, \\
A(m_{12}) &= \frac{im_{23}(m_{13} + j_1j_2j_3)}{m_{12}(m_{12} + j_2j_3)} \quad (42)
\end{aligned}$$

$$D(m_{12}) = \frac{1}{m_{12}}\sqrt{\frac{[(m_{13} + j_1j_2j_3)^2 - j_1^2m_{12}^2](m_{12}^2 - j_3^2m_{23}^2)}{4m_{12}^2 - j_2^2j_3^2}}$$

The rest of the generators are obtained by the commutators

$$X_{13} = [X_{23}, X_{12}], \quad X_{02} = [X_{12}, X_{01}], \quad X_{03} = [X_{23}, X_{02}]. \quad (43)$$

For completeness we give the explicit expressions of these generators,

$$\begin{aligned}
X_{02}|m\rangle &= \frac{1}{2}A(m_{12}) \left[a(m_{11} + j_3) \left| \begin{matrix} m_{12} \\ m_{11} + j_3 \end{matrix} \right\rangle + \right. \\
&+ a(m_{11} + j_3) \left| \begin{matrix} m_{12} \\ m_{11} + j_3 \end{matrix} \right\rangle \left. \right] - \frac{1}{2j_3}D(m_{12}) \\
&\times \left[\sqrt{(m_{12} - j_2m_{11} - j_2j_3)(m_{12} - j_2m_{11})} \left| \begin{matrix} m_{12} - j_2j_3 \\ m_{11} + j_3 \end{matrix} \right\rangle - \right. \\
&- \sqrt{(m_{12} + j_2m_{11} - j_2j_3)(m_{12} + j_2m_{11})} \left| \begin{matrix} m_{12} - j_2j_3 \\ m_{11} - j_3 \end{matrix} \right\rangle \left. \right] - \frac{1}{2j_3}D(m_{12} + j_2j_3) \\
&\times \left[\sqrt{(m_{12} + j_2m_{11} + j_2j_3)(m_{12} + j_2m_{11} + 2j_2j_3)} \left| \begin{matrix} m_{12} + j_2j_3 \\ m_{11} + j_3 \end{matrix} \right\rangle - \right. \\
&- \sqrt{(m_{12} - j_2m_{11} + j_2j_3)(m_{12} - j_2m_{11} + 2j_2j_3)} \left| \begin{matrix} m_{12} + j_2j_3 \\ m_{11} - j_3 \end{matrix} \right\rangle \left. \right] \quad (44) \\
X_{03}|m\rangle &= -j_3\frac{i}{2}A(m_{12}) \left[a(m_{11} + j_3) \left| \begin{matrix} m_{12} \\ m_{11} + j_3 \end{matrix} \right\rangle - \right. \\
&- a(m_{11} + j_3) \left| \begin{matrix} m_{12} \\ m_{11} + j_3 \end{matrix} \right\rangle \left. \right] + \frac{i}{2}D(m_{12}) \\
&\times \left[\sqrt{(m_{12} - j_2m_{11} - j_2j_3)(m_{12} - j_2m_{11})} \left| \begin{matrix} m_{12} - j_2j_3 \\ m_{11} + j_3 \end{matrix} \right\rangle + \right. \\
&+ \sqrt{(m_{12} + j_2m_{11} - j_2j_3)(m_{12} + j_2m_{11})} \left| \begin{matrix} m_{12} - j_2j_3 \\ m_{11} - j_3 \end{matrix} \right\rangle \left. \right] + \frac{i}{2j_3}D(m_{12} + j_2j_3) \\
&\times \left[\sqrt{(m_{12} + j_2m_{11} + j_2j_3)(m_{12} + j_2m_{11} + 2j_2j_3)} \left| \begin{matrix} m_{12} + j_2j_3 \\ m_{11} + j_3 \end{matrix} \right\rangle + \right. \\
&+ \sqrt{(m_{12} - j_2m_{11} + j_2j_3)(m_{12} - j_2m_{11} + 2j_2j_3)} \left| \begin{matrix} m_{12} + j_2j_3 \\ m_{11} - j_3 \end{matrix} \right\rangle \left. \right].
\end{aligned}$$

Obviously the generator X_{13} is obtained from the generator X_{02} (29) , with the replacement of the parameters j_1, j_2 by j_2, j_3 , correspondingly.

The generators (42) are realized in the basis, which is characterized by the chain of subalgebras $so_4(j) \supset so_3(j_2, j_3) \supset so_2(j_3)$, where $so_3(j_2, j_3) = \{X_{12}, X_{13}, X_{23}\}$, $so_2(j_3) = \{X_{23}\}$.

The $j_1 = \iota_1$ contracted algebra has the structure of the semidirect sum and is the inhomogeneous algebra

$$so(4; \iota_1, j_2, j_3) = \{X_{01}, X_{02}, X_{03}\} (+so(3; j_2, j_3) \cong iso(3; j_2, j_3). \quad (45)$$

The highest weight components are transformed as follows; $k \equiv m_{13} = \iota_1 j_2 j_3 m_{13}^*$, $k \in \mathbf{R}$, $m_{23} = j_2 m_{23}^*$ and the generators are given by Eqs. (44) with

$$A(m_{12}) = im_{13} m_{23} / m_{12} (m_{12} + j_2 j_3) \equiv ikm_{23} / m_{12} (m_{12} + j_2, j_3). \quad (46)$$

The eigenvalues of the Casimir operators are $C_2(\iota_1) = k^2$, $C'_2(\iota_1) = -km_{23}$.

The $j_3 = \iota_3$ contracted algebra has the structure of the semidirect sum and is also the inhomogeneous algebra

$$so(4; j_1, j_2, \iota_3) = \{X_{03}, X_{13}, X_{23}\} (+so(3; j_1, j_2) \cong iso(3; j_1, j_2). \quad (47)$$

The component of $|m\rangle$ are transformed as follows: $k \equiv m_{13} = j_1 j_2 \iota_3 m_{13}^*$, $m_{23} = j_2 m_{23}^*$, $p \equiv m_{12} = j_2 \iota_3 m_{12}^*$, $q \equiv m_{11} = \iota_3 m_{11}^*$, $k, p, q \in \mathbf{R}$ and from Eqs.(44) we obtain the generators of the contracted algebra $so_4(j_1, j_2, \iota_3)$ in the form

$$\begin{aligned} X_{23}|m\rangle &= iq|m\rangle, \quad |m\rangle = \begin{vmatrix} k & m_{23} \\ p & \\ q & \end{vmatrix}, \\ X_{12}|m\rangle &= \sqrt{p^2 - j_2^2 q^2} \frac{\partial|m\rangle}{\partial q} - \frac{j_2^2 q}{2\sqrt{p^2 - j_2^2 q^2}} |m\rangle, \\ X_{13}|m\rangle &= i\sqrt{p^2 - j_2^2 q^2} |m\rangle, \\ X_{01}|m\rangle &= \frac{ikqm_{23}}{p^2} |m\rangle + \frac{1}{2p} \sqrt{(p^2 - j_2^2 q^2)(k^2 - j_1^2 p^2)} \\ &\times \left\{ 2 \frac{\partial|m\rangle}{\partial p} + \frac{p}{p^2 - j_2^2 q^2} |m\rangle - \frac{k^2}{p(k^2 - j_1^2 p^2)} |m\rangle \right\}, \\ X_{02}|m\rangle &= \frac{ikm_{23}}{p^2} \sqrt{p^2 - j_2^2 q^2} |m\rangle + \frac{1}{2p} \sqrt{k^2 - j_1^2 p^2} \\ &\times \left\{ 2p \frac{\partial|m\rangle}{\partial q} + 2j_2^2 q \frac{\partial|m\rangle}{\partial p} - j_2^2 \frac{qk^2}{p(k^2 - j_1^2 p^2)} |m\rangle \right\}, \\ X_{03}|m\rangle &= i\sqrt{k^2 - j_1^2 p^2} |m\rangle. \end{aligned} \quad (48)$$

The contracted algebra $so_4(\iota_1, 1, 1)$ is isomorphic with the algebra $so_4(1, 1, \iota_3)$ as long as they are both isomorphic with the inhomogeneous algebra iso_3 . Therefore the generators (44) and (46) for $j_2 = j_3 = 1$ realize the representation of the inhomogeneous algebra iso_3 in the discrete basis, which is characterized by the chain of subalgebras $iso_3 \supset so_3 \supset so_2$, and the generators (48)

for $j_1 = j_2 = 1$ realize the same representation of the iso_3 in the continuous basis, which is characterized by the chain $iso_3 \supset so(2, 1) \supset so(1, 1)$.

This example illustrates the general situation: if different contractions give isomorphic algebras, then we obtain the representation of the contracted algebra in different basis. Let us also note that contractions as well as analytic continuations are the methods of the construction of the new representations from the well-known representations of the classical algebras.

The last one-dimensional contraction of the algebra so_4 is achieved when the parameter j_2 is equal to the dual unit. The generators of the contracted algebra $so_4(j_1, \iota_2, j_3)$ are easily obtained from Eqs.(44) for $j_2 = \iota_2$ in the form

$$\begin{aligned}
X_{01}|m\rangle &= \frac{iksm_{11}}{p^2}|m\rangle + f(k, s, p) \left\{ 2 \frac{\partial|m\rangle}{\partial p} + j_3^2 \frac{s^2}{p(p^2 - j_3^2 s^2)}|m\rangle - j_1^2 \frac{p^2}{k^2 - j_1^2 p^2}|m\rangle \right\}, \\
X_{02}|m\rangle &= \frac{iks}{2p} \left(\left| \begin{array}{c} p \\ m_{11} + j_3 \end{array} \right\rangle + \left| \begin{array}{c} p \\ m_{11} - j_3 \end{array} \right\rangle \right) - \\
&\quad - \frac{1}{j_3} f(k, s, p) \left(\left| \begin{array}{c} p \\ m_{11} + j_3 \end{array} \right\rangle - \left| \begin{array}{c} p \\ m_{11} - j_3 \end{array} \right\rangle \right), \\
X_{03}|m\rangle &= \frac{iks}{2p} \left(\left| \begin{array}{c} p \\ m_{11} + j_3 \end{array} \right\rangle - \left| \begin{array}{c} p \\ m_{11} - j_3 \end{array} \right\rangle \right) + \frac{1}{j_3} f(k, s, p) \\
&\quad \times \left(\left| \begin{array}{c} p \\ m_{11} + j_3 \end{array} \right\rangle + \left| \begin{array}{c} p \\ m_{11} - j_3 \end{array} \right\rangle \right), \\
X_{12}|m\rangle &= \frac{m_{11}}{2j_3} \left(\left| \begin{array}{c} p \\ m_{11} + j_3 \end{array} \right\rangle - \left| \begin{array}{c} p \\ m_{11} - j_3 \end{array} \right\rangle \right), \\
X_{13}|m\rangle &= \frac{i}{2} m_{11} \left(\left| \begin{array}{c} p \\ m_{11} + j_3 \end{array} \right\rangle + \left| \begin{array}{c} p \\ m_{11} - j_3 \end{array} \right\rangle \right), \\
X_{23}|m\rangle &= im_{11}|m\rangle, \\
f(k, s, p) &= \frac{1}{2p} \sqrt{(k^2 - j_1^2 p^2)(p^2 - j_3^2 s^2)}, \quad |m\rangle = \left| \begin{array}{cc} k & s \\ p & \\ m_{11} & \end{array} \right\rangle. \quad (49)
\end{aligned}$$

The eigenvalues of the Casimir operators are as follows:

$$C_2(\iota_2) = k^2 + j_1^2 j_3^2 s^2, \quad C_2'(\iota_2) = -ks. \quad (50)$$

The generators of the maximal contracted algebra $so(4; \iota)$ are given by Eqs.(44) for $j_1 = \iota_1$, $j_2 = \iota_2$, $j_3 = \iota_3$, or by Eqs.(48) for $j_1 = \iota_1$, $j_2 = \iota_2$ in the form

$$\begin{aligned}
X_{23}|m\rangle &= iq|m\rangle, \quad |m\rangle = \left| \begin{array}{cc} k & s \\ p & \\ q & \end{array} \right\rangle, \\
X_{12}|m\rangle &= p \frac{\partial|m\rangle}{\partial q}, \quad X_{13}|m\rangle = iq|m\rangle, \\
X_{01}|m\rangle &= \frac{iks q}{p^2}|m\rangle + k \frac{\partial|m\rangle}{\partial p}, \quad X_{03}|m\rangle = ik|m\rangle,
\end{aligned} \quad (51)$$

$$X_{02}|m\rangle = \frac{iks}{p}|m\rangle - k\frac{\partial|m\rangle}{\partial q}, \quad k \geq 0, p \geq 0, s, q \in \mathbf{R},$$

and the Casimir operators have the following nonzero eigenvalues: $C_2(\iota) = k^2$, $C'_2(\iota) = -ks$.

The developed method for imaginary values of the parameters j gives the explicit expressions of the generators of the representations of the pseudoorthogonal algebras with a different signature. In particular, Eqs. (44) for $j_1 = i$, $j_2 = j_3 = 1$ describe the irreducible representations of the Lorentz algebra $so(1, 3)$ (compare with Ref. 20, Chap. 19, Sec. 7).

5 Conclusion

In previous sections we have developed the simple but effective method of the uniform treatment of all orthogonal Cayley–Klein algebras and their representations. It follows from this treatment that it is not necessary to investigate each Cayley–Klein algebra separately. Almost all their algebraic properties are obtained by corresponding transformations of the well-known properties of the classical orthogonal algebras. The same nature of the two previously different operations in group theory, namely, the Wigner-Inonu contraction and the Weyl unitary trick is clearly displayed in this manner. Such uniform treatment of all Cayley–Klein algebras instead of only one algebra gives us the opportunity to display the general properties that are valid for the whole set of the algebras and therefore are more fundamental ones.

We have regarded the Cayley–Klein groups and algebras starting from the geometrical picture. Such an approach provides not only clear interpretation but also knowing from the very beginning the resulting contracted algebra, unlike the usual approach to the contraction. Indeed, in the usual approach contraction is defined as a singular transformation of generators [2] or a singular transformation in the space of eigenvalues [21] and the contracted generators and their commutation relations are known only after the limiting procedure.

In the case when different contractions and analytic continuations give isomorphic algebras the developed approach gives the representations of the contracted and continued algebras in different bases, which correspond to the different algebra reductions. Therefore it is possible to obtain not only the representations of the new algebras but also the representations in different bases.

The law of transformations (18) of the components of the Gel'fand–Tsetlin patterns have been chosen in such a way that the eigenvalues of all Casimir operators differ from zero for all contractions. We name such transformations as general transformations. Apart from the general transformations there are many transformations of the components $|m\rangle$ that give the representations of $so_{n+1}(\mathbf{j})$ with zero eigenvalues of all or some Casimir operators under contractions. In other words, we may obtain the different degenerate representations of the contracted algebras from the Gel'fand–Tsetlin representation of the classical algebra. For example, it is possible not to transform the components of the patterns $m = m^*$; then the eigenvalues of all Casimir operators will be equal to zero when all parameters $\mathbf{j}$ are equal to dual units.

It is also possible to obtain the representations of the algebra $so_{k+1}(\mathbf{j}')$, $\mathbf{j}' = (j_1, \dots, j_k)$ from the representations of the algebra $so_{n+1}(\mathbf{j})$ with $k < n$. As directly follows from Eqs. (1), this will be achieved when all parameters j_i

with $k < l \leq n$ are equal to the same dual unit ι_{k+1} , i.e., $j_{k+1} = \iota_{k+1}$, $j_{k+2} = \iota_{k+1} \dots, j_n = \iota_{k+1}$. Then all coordinates X_l , $K < l \leq n$ are identically equal to zero and the map ψ gives in result the Cayley–Klein space with a smaller dimensions.

This approach is also applicable for quantum algebras [12], [22]. In the last case the unified description is split into two different methods: one for the contractions and another for the analytic continuations of the quantum algebras.

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