IRREDUCIBLE REPRESENTATIONS
OF CAYLEY-KLEIN UNITARY
ALGEBRAS

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Abstract

Multidimensional contractions of irreducible representations of the Cayley-Klein unitary algebras in the Gel’fand-Zetlin basis are considered. Contracted over different parameters, algebras can turn out to be isomorphic. In this case method of transitions describes the same reducible representations in different bases, say, discrete and continuous ones.
1 Representation of unitary algebras \( u(2; j_1) \)

1.1 Finite-dimensional irreducible representations of algebra \( u(2) \)

These representations have been described by Gel’fand and Zetlin [1]. They are realized in the space with orthogonal basis, determined by a scheme with integer-valued components

\[
|m^*\rangle = \begin{pmatrix}
m^*_{12} & m^*_{11} & m^*_{22}
\end{pmatrix}
\quad m^*_{12} \geq m^*_{11} \geq m^*_{22},
\]

by operators

\[
E^*_{11}|m^*\rangle = m^*_{11}|m^*\rangle = A^*_{11}|m^*\rangle,
\]

\[
E^*_{22}|m^*\rangle = (m^*_{12} + m^*_{22} - m^*_{11})|m^*\rangle = A^*_{00}|m^*\rangle,
\]

\[
E^*_{21}|m^*\rangle = \sqrt{(m^*_{12} - m^*_{11} + 1)(m^*_{11} - m^*_{22})}|m^*\rangle = A^*_{01}|m^*\rangle,
\]

\[
E^*_{12}|m^*\rangle = \sqrt{(m^*_{12} - m^*_{11})(m^*_{11} + 1 - m^*_{22})}|m^*\rangle = A^*_{10}|m^*\rangle,
\]

where \( |m^*_{11} \pm 1\rangle \) means the scheme (1) with the component \( m^*_{11} \) changed for \( m^*_{11} \pm 1 \).

Let us change standard notations of generators \( E_{kr} \) for new notations \( A_{n-k,n-r} \), \( n = 2 \), consistent with the notations of [2]. The irreducible representation is completely determined by the components \( m^*_{12}, m^*_{22}, m^*_{12} \geq m^*_{22} \) of the upper row in (1) (components of the major weight).

As it is known, Casimir operators are proportional to the unit operators on the space of irreducible representation. The spectrum of Casimir operators for classical groups was found in [3]-[5]; for semisimple groups and algebra \( u(2) \) it is as follows:

\[
C^*_1 = m^*_{12} + m^*_{22}, \quad C^*_2 = m^*_{12}^2 + m^*_{22}^2 + m^*_{12} - m^*_{22}.
\]

Let us remind that the asterisk marks the quantities referring to the classical groups (algebras).

In the space of representation there is vector of the major weight \( \varphi_{Mw} \), described by the scheme (1) for \( m^*_{11} = m^*_{12} \). Acting on it, the rising operator \( A^*_{10} \) gives zero (\( A^*_{10}\varphi_{Mw} = 0 \)) and the lowering operator \( A^*_{01} \) makes the value \( m^*_{11} = m^*_{12} \) less by one (\( A^*_{01}\varphi_{Mw} = \sqrt{m^*_{12} - m^*_{22}|m^*_{12} - 1\rangle} \)). Consequently applying \( A^*_{01} \) to \( \varphi_{Mw} \), we come to the vector of the minor weight \( \varphi_{mw} \), described by the scheme (1) for \( m^*_{11} = m^*_{22} \). Acting on \( \varphi_{mw} \), the lowering operator gives zero (\( A^*_{01}\varphi_{mw} = 0 \)).

The irreducible representations is finite-dimensional, and this fact is reflected in the inequalities (ref1), which are satisfied by the component \( m^*_{11} \) of the scheme.

The condition of unitary for representations of the algebra \( u_2 \) is equivalent to the following relations for the operators (3): \( A^*_{kk} = \bar{A}^*_{kk} \) \( (k = 0, 1) \), \( A^*_{01} = \bar{A}^*_{10} \), where the bar means the complex conjugation. For matrix elements the conjugation of unitary can be written as follows

\[
\langle m^*|A^*_{00}|m^*\rangle = \overline{\langle m^*|A^*_{00}|m^*\rangle}, \quad \langle m^*|A^*_{11}|m^*\rangle = \overline{\langle m^*|A^*_{11}|m^*\rangle},
\]

\[
\langle m^*_{11} - 1|A^*_{01}|m^*\rangle = \overline{\langle m^*|A^*_{10}|m^*_{11} - 1\rangle}.
\]
1.2 Transition to the representations of algebra \( u(2; j_1) \)

Under transition from the algebra \( u(2) \) to the algebra \( u(2; j_1) \) the generators \( A_{00}^\ast, A_{11}^\ast \) and the Casimir operators \( C_1^\ast \) remain unchanged, and the generators \( A_{01}^\ast, A_{10}^\ast \) and the Casimir operator \( C_2^\ast \) are transformed as follows (see [2]):

\[
A_{01}^\ast = j_1 A_{01}^\ast(-), \quad A_{10}^\ast = j_1 A_{10}^\ast(-),
\]

\[
C_2(j_1) = j_1^2 C_2^\ast(-),
\]

where \( A_{01}^\ast(-), A_{10}^\ast(-) \) are singularly transformed (for dual value of parameter \( j_1 = \iota_1 \)) generators of the initial algebra \( u(2) \). A question is now raised as to how to set this transformation for the irreducible representation (2) of the algebra \( u(2) \). Let us give the transformation of the components of scheme (1) as follows:

\[
m_{12} = j_1 m_{12}^\ast, \quad m_{22} = j_1 m_{22}^\ast, \quad m_{11} = m_{11}^\ast.
\]

(6)

Then, taking into account (5), the representation generators (2) can be written as

\[
A_{00}|m\rangle = \left(\frac{m_{12} + m_{22}}{j_1} - m_{11}\right)|m\rangle, \quad A_{11}|m\rangle = m_{11}|m\rangle,
\]

\[
A_{01}|m\rangle = \sqrt{(m_{12} - j_1(m_{11} - 1))(j_1 m_{11} - m_{22})} m_{11} - 1),
\]

\[
A_{10}|m\rangle = \sqrt{(m_{12} - j_1 m_{11})[j_1 (m_{11} + 1) - m_{22}] m_{11} + 1},
\]

(7)

and the spectrum of Casimir operators

\[
C_1(j_1) = A_{00} + A_{11},
\]

\[
C_2(j_1) = A_{01} A_{10} + A_{10} A_{01} + j_1^2 (A_{00}^2 + A_{11}^2)
\]

(8)

are

\[
C_1(j_1) = \frac{m_{12} + m_{22}}{j_1},
\]

\[
C_2(j_1) = m_{12}^2 + m_{22}^2 + j_1 (m_{12} - m_{22}),
\]

(9)

where \( |m\rangle \) means the following scheme

\[
|m\rangle = \left| \begin{array}{cc} m_{12} & m_{22} \\ m_{11} & \end{array} \right|
\]

(10)

The inequality (1) for components can be formally written as

\[
\frac{m_{12}}{j_1} \geq m_{11} \geq \frac{m_{22}}{j_1}, \quad \frac{m_{12}}{j_1} \geq \frac{m_{22}}{j_1}.
\]

(11)

To reveal the sense of these inequalities for \( j_1 \neq 1 \), we shall discuss the action of rising operator \( A_{10} \) on the vector of the “major weight” \( \varphi_{Mw} \), described by scheme (10) for \( m_{11} = m_{12} \), and the lowering operator \( A_{01} \) on the vector of the “minor weight” \( \varphi_{mw} \), described by scheme (10) for \( m_{11} = m_{22} \). We obtain

\[
A_{10} \varphi_{Mw} = \sqrt{m_{12}(1 - j_1)(j_1 (m_{12} + 1) - m_{22})} m_{12} + 1),
\]

3
It can be seen from here that for \( j_1 = \iota_1, i \) these expressions differ from zero. Therefore, the space of representation is infinite-dimensional, and the integer-valued component \( m_{11} \), which numbers the basis vectors, varying from \(-\infty\) to \( \infty \). Thus, the formal inequalities (11) for \( j_1 = \iota_1, i \) are interpreted as \( \infty > m_{11} > -\infty \) and \( m_{12} \geq m_{22} \).

The form (6) of the transformation of Gel’fand-Zetlin scheme is chosen in such a way that the Casimir operator of the second order would differ from zero and not contain indeterminate expressions for \( j_1 = \iota_1 \).

### 1.3 Contraction of irreducible representations

For \( j_1 = \iota_1 \) the operator \( A_{00} \) contains the summand \( (m_{12} + m_{22})/\iota_1 \), which is in general, undeterminate, if its numerator is a real, complex or dual number. This summand is determinate, if its numerator is purely dual number (see [6]) \( m_{12} + m_{22} = \iota_1 \zeta \), where \( \zeta \in R \) or \( \zeta \in C \). The requirement of unitarity for the operator \( A_{00} \) is given by \( \zeta \in R \). Thus, in order that the operators (7) would determine the representation of the algebra \( u(2; \iota_1) \), it is necessary to choose the components \( m_{12}, m_{22} \) of the scheme (9) as follows

\[
m_{12} = k + \iota_1 \zeta/2, \quad m_{22} = -k + \iota_1 \zeta/2, \quad \zeta \in R,
\]

(13)

where \( k \), generally speaking, is a complex number.

The scheme (10) for dual values of components is determined by the expansion into series

\[
|m\rangle = \begin{pmatrix} k + \iota_1 \zeta/2 \\ m_{11} \end{pmatrix} - \begin{pmatrix} -k + \iota_1 \zeta/2 \\ m_{11} \end{pmatrix} = \hat{m} + \iota_1 \frac{\zeta}{2} (|\hat{m}\rangle'_{12} + |\hat{m}\rangle'_{22}),
\]

where

\[
|\hat{m}\rangle = \begin{pmatrix} k \\ m_{11} \end{pmatrix} - \begin{pmatrix} -k \\ m_{11} \end{pmatrix}
\]

and the similar expressions are valid for \( |\hat{m}\rangle'_{22} \). The initial schemes (1) are normalized to unit:

\[
\langle m^* | m^* \rangle = \delta_{m_{11}' m_{11}'} \delta_{m_{22}' m_{22}'} \delta_{\iota_1 m_{11}},
\]

Schemes (10) for the continuous values of components are normalized to delta-function. In particular, for \( |\hat{m}\rangle \) we have normalization to the squared delta-function

\[
\langle \hat{m} | \hat{m} \rangle = \delta^2 (k' - k) \delta_{m_{11}' m_{11}}.
\]

(14)

Substituting (13), (??) in formulas of §1.2, we obtain the representation operators of the algebra \( u(2; \iota_1) \) (the dual parts are omitted):

\[
A_{00} |\hat{m}\rangle = (\zeta - m_{11}) |\hat{m}\rangle, \quad A_{11} |\hat{m}\rangle = m_{11} |\hat{m}\rangle,
\]

\[
A_{01} |\hat{m}\rangle = k |m_{\infty} - 1\rangle, \quad A_{10} |\hat{m}\rangle = k |m_{\infty} + 1\rangle.
\]

(15)

The requirement of unitarity (4) for operators \( A_{01}, A_{10} \) gives \( k = \bar{k} \), i.e. \( k \) is a real number, the inequality \( m_{12} \geq m_{22} \) gives for the real parts \( k \geq -k \), i.e. \( k \geq 0 \), the component \( m_{11} \) is integer-valued and changes according to (10) in the range \(-\infty < m_{11} < \infty \). The eigenvalues of Casimir operators (8) on the irreducible representations of the algebra \( u(2; \iota_1) \) are

\[
C_1(\iota_1) = \zeta, \quad C_2(\iota_1) = 2k^2.
\]

(16)
They are independent and differ from zero. As in the case of the initial algebra $u(2)$, the irreducible representations of the contracted algebra $u(2; \iota_1)$ are completely determined by the upper row of the scheme, i.e. by parameters $k \geq 0$, $\zeta \in R$. The results (7), (14) coincide with the corresponding formulas in [7] for the case of algebra $iu(1)$.

To the requirement of determinacy of the spectrum of operator $C_2(\iota_1)$ corresponds not only the transformation (6) of the components of Gel’fand-Zetlin schemes, but, for example, the transformation $m_{12} = j_1 m_{12}^*$, $m_{22} = m_{22}^*$, $m_{11} = m_{11}^*$ as well. Here generator $A_{00}|m\rangle = (m_{12} + m_{22} - m_{11})|m\rangle$ is determined only for $m_{12} = \iota_1 p$, $p \in R$, but then $C_1(\iota_1) = p + m_{22} \neq 0$, and $C_2(\iota_1) = \iota_1^2 [m_{22}(m_{22} - 1) + m_{12}^2/\iota_1^2 + m_{12}/\iota_1] = m_{12}^2 + \iota_1 m_{12} = (\iota_1 p)^2 + \iota_1 (\iota_1 p) = 0$. In this case the irreducible representation of the algebra $u(2)$ is contracted to the degenerate representation of the algebra $u(2; \iota_1)$, for which $C_1(\iota_1) \neq 0$, and $C_2(\iota_1) = 0$. One can not, at all, transform the components $m_{kr} = m_{kr}^*$. Then under contraction we also obtain the degenerate representation of the algebra $u(2; \iota_1)$ with $C_1(\iota_1) = m_{12} + m_{22} \neq 0$, $C_2(\iota_1) = 0$. This representation is given by generators $A_{00}$, $A_{11}$ of the form (2), and the generators $A_{01}$ and $A_{10}$ bring $|m\rangle$ to zero: $A_{01}|m\rangle = 0, A_{10}|m\rangle = 0$.

We have chosen the transformation (6) which gives under contraction the non-degenerated general representation of the algebra $u(2; \iota_1)$ with non-zero spectrum of all Casimir operators. Further, studying the algebras of the higher dimensions, we shall consider just this case.

### 1.4 Analytical continuation of irreducible representations

As it has been noticed in [6], the formulas for the transformation of algebraic quantities, derived from the requirement of the absence of undeterminate expressions for dual values of parameters $j$, are valid for imaginary values of parameters as well. For the algebra $u(2; j_1 = i) \equiv u(1, 1)$ this means that $(m_{12} + m_{22})/i = \zeta$. The requirement of unitarity for $A_{00}$ gives $\zeta \in R$, i.e. components $m_{12}$ and $m_{22}$, in general, are

$$m_{12} = a + i\left(\frac{b + \zeta}{2}\right), \quad m_{22} = -a - i\left(\frac{b - \zeta}{2}\right), \quad a, b, \zeta \in R.$$  \hspace{1cm} (17)

Substituting (17) in (9), (7), we get

$$A_{00}|m\rangle = (\zeta - m_{11})|m\rangle, \quad A_{11}|m\rangle = m_{11}|m\rangle,$$

$$A_{01}|m\rangle = \sqrt{a^2 - b(b + 1) + \left(\frac{\zeta}{2} - m_{11}\right) \left(\frac{\zeta}{2} - m_{11} + 1\right) + ia(2b + 1)m_{11} - 1},$$

$$A_{10}|m\rangle = \sqrt{a^2 - b(b + 1) + \left(\frac{\zeta}{2} - m_{11}\right) \left(\frac{\zeta}{2} - m_{11} - 1\right) + ia(2b + 1)m_{11} + 1}$$  \hspace{1cm} (18)

$$C_1(i) = \zeta, \quad C_2(i) = 2 \left[a^2 - b(b + 1) - \left(\frac{\zeta}{2}\right)^2\right] + 2ia(2b + 1).$$

The relation (4) for the operators $A_{01}$, $A_{10}$, implied by the requirement of Hermiticity, can be written as follows

$$\sqrt{a^2 - b(b + 1) + \left(\frac{\zeta}{2} - m_{11}\right) \left(\frac{\zeta}{2} - m_{11} + 1\right) + ia(2b + 1)^2} =$$
\[
\sqrt{a^2 - b(b + 1) + \left(\frac{\zeta}{2} - m_{11}\right)\left(\frac{\zeta}{2} - m_{11} + 1\right)} - ia(2b + 1). \tag{19}
\]

To satisfy (19) for any \(\zeta, m_{11}\), the imaginary part of the radicand must vanish and the real part must be positive. It is possible in two cases: \(a\) \(b = -\frac{1}{2}, a \neq 0; \ b\) \(a = 0, -b(b + 1) > 0.\)

In the case \((a)\) the formulas (18) can be rewritten as follows:

\[
A_{01}|m\rangle = \sqrt{a^2 + [m_{11} - (1 - \zeta)/2]^2}|m_{11} - 1\rangle,
\]

\[
A_{10}|m\rangle = \sqrt{a^2 + [m_{11} + (1 - \zeta)/2]^2}|m_{11} + 1\rangle,
\]

\[
C_1(i) = \zeta, \quad C_2(i) = 2a^2 + (1 - \zeta^2)/2. \tag{20}
\]

This is irreducible representation of the continuous series of the algebra \(u(1, 1)\). Gel’fand and Graev [8] used the components \(\tilde{m}_{12} = -\frac{1}{2} + \sigma, \tilde{m}_{22} = \frac{1}{2} + \sigma,\) related with components \(m_{12} = i\tilde{m}_{12}, m_{22} = i\tilde{m}_{22}\) via formulas \(\text{Re}\sigma = \frac{\zeta}{2}, \text{Im}\sigma = -a.\)

In the case \((b)\) the relations (18) can be rewritten as follows:

\[
A_{01}|m\rangle = \sqrt{|m_{11} - (\zeta + 1)/2|^2 - \left(b + \frac{1}{2}\right)^2}|m_{11} - 1\rangle,
\]

\[
A_{10}|m\rangle = \sqrt{|m_{11} - (\zeta - 1)/2|^2 - \left(b + \frac{1}{2}\right)^2}|m_{11} + 1\rangle,
\]

\[
C_1(i) = \zeta, \quad C_2(i) = -2\left[b(b + 1) + \frac{1}{4}\zeta^2\right]. \tag{21}
\]

This is irreducible representation of additional continuous series [9].

There is once more possibility besides cases \((a)\) and \((b)\). Let components \(m_{12}, m_{22}\) be purely imaginary: \(m_{12} = i\tilde{m}_{12}, m_{22} = i\tilde{m}_{22},\) where \(\tilde{m}_{12}, \tilde{m}_{22}\) are integers. Then the relations (18) can be rewritten as follows:

\[
A_{00}|m\rangle = (\tilde{m}_{12} + \tilde{m}_{22} - m_{11})|m\rangle, \quad A_{11}|m\rangle = m_{11}|m\rangle,
\]

\[
A_{01}|m\rangle = \sqrt{-(\tilde{m}_{12} - m_{11} + 1)(m_{11} - \tilde{m}_{22})}|m_{11} - 1\rangle,
\]

\[
A_{10}|m\rangle = \sqrt{-(\tilde{m}_{12} - m_{11})(m_{11} + 1 - \tilde{m}_{22})}|m_{11} + 1\rangle,
\]

\[
C_1(i) = \tilde{m}_{12} + \tilde{m}_{22}, \quad C_2(i) = -(\tilde{m}_{12}^2 + \tilde{m}_{22}^2 + \tilde{m}_{12} - \tilde{m}_{22}). \tag{22}
\]

They coincide with (2), (3) except for the sign minus in the radicand. The requirement of unitarity (4) can be reduced to the reality of the root in the expressions for the generators \(A_{01}\) and \(A_{10}\) which is possible when one of the factors is negative. As a result, we get two more irreducible representations: \(c\) \(m_{11} \geq \tilde{m}_{12} + 1; \ d\) \(m_{11} \leq \tilde{m}_{22} - 1,\) which are called discrete series. The discrete series of irreducible representations of pseudounitary algebras \(u(p, q)\) are described by Gel’fand and Graev [8], [10]. The cases \((c)\) and \((d)\) correspond to modified schemes

\[
\begin{pmatrix}
  m_{11} & \tilde{m}_{12} & \tilde{m}_{22} \\
  m_{11} & \tilde{m}_{12} & \tilde{m}_{22} \\
\end{pmatrix}.
\tag{23}
\]

In the simplest case of algebras \(u(2; j_1)\) we have shown in detail how method of transitions works for irreducible representations.
The irreducible representations of algebras $u(2; j_1)$ are given by formulas of §1.2 with additional conditions (13) in the case of contraction and (17) in the case of analytical continuation to the components of the upper row in Gel’fand-Zetlin scheme. To obtain unitary representation, it is necessary additionally to check up whether the relations (4) are satisfied for contracted and analytically continued generators of representation.

2 Representations of unitary algebras $u(3; j_1, j_2)$

2.1 Description of representations

Standard notations of Gel’fand and Zetlin [11] correspond to diminishing chain of subalgebras $u(3) \supset u(2) \supset u(1)$, where $u(3) = \{E_{kr}, k, r = 1, 2, 3\}$; $u(2) = \{E_{kr}, k, r = 1, 2\}$; $u(1) = \{E_{11}\}$. To make them consistent with the notations of [2], it is necessary to change index $k$ for index $n - k = 3 - k$, i.e. $E_{kr} = A_{n-k,n-r}$. Doing so, we turn the chain of subalgebras into $u(3; j_1, j_2) \supset u(2; j_2) \supset u(1)$, where $u(3; j) = \{A_{sp}, s, p = 0, 1, 2\}$; $u(2; j_2) = \{A_{sp}, s, p = 1, 2\}$; $u(1) = \{A_{22}\}$. The component enumeration in Gel’fand-Zetlin schemes we leave unchanged.

It is well known that to determine representations of algebra $u(3)$ it is sufficient to determine action of generators $E_{pp}$, $E_{p-p} = E_{p-1,p}$, i.e. generators $A_{kk}$ ($k = 0, 1, 2$), $A_{k,k+1}$, $A_{k+1,k}$ ($k = 0, 1$). The rest generators $A_{02}$, $A_{20}$ can be found using commutators $A_{02} = [A_{01}, A_{12}]$, $A_{20} = [A_{21}, A_{10}]$. Under transition from $u(3)$ to $u(3; j)$ the generators are transformed as follows (see [2]: $A_{01} = j_1 A_{01}^* (\rightarrow)$, $A_{12} = j_2 A_{12}^* (\rightarrow)$, $A_{10} = j_1 A_{10}^* (\rightarrow)$, $A_{21} = j_2 A_{21}^* (\rightarrow)$, $A_{kk} = A_{kk}^* (\rightarrow)$). Transformation of the components of Gel’fand-Zetlin schemes can be defined as follows:

\[
m_{13} = j_1 j_2 m_{13}^*, \quad m_{23} = m_{23}^*, \quad m_{33} = j_1 j_2 m_{33}^*, \quad m_{12} = j_2 m_{12}^*, \quad m_{22} = j_2 m_{22}^*, \quad m_{11} = m_{11}^*.
\]  

(24)

Then the component of scheme $|m\rangle$ satisfy inequalities

\[
|m\rangle = \begin{pmatrix} m_{13} & m_{23} & m_{33} \\ m_{12} & m_{22} & m_{11} \end{pmatrix},
\]

\[
\frac{m_{13}}{j_1 j_2} \geq \frac{m_{23}}{j_1 j_2} \geq \frac{m_{33}}{j_1 j_2}, \quad \frac{m_{13}}{j_1 j_2} \geq \frac{m_{12}}{j_2} \geq \frac{m_{23}}{j_2},
\]

\[
m_{23} \geq \frac{m_{23}}{j_2} \geq \frac{m_{33}}{j_1 j_2}, \quad \frac{m_{12}}{j_2} \geq \frac{m_{11}}{j_2} \geq \frac{m_{22}}{j_2}.
\]

(25)

Transforming the known expressions for generators of algebra $u(3)$ we come to generators of representations of algebra $u(3; j)$:

\[
A_{00}|m\rangle = \left( m_{23} + \frac{m_{13} + m_{33}}{j_1 j_2} - \frac{m_{12} + m_{22}}{j_2} \right) |m\rangle,
\]

\[
A_{01}|m\rangle = \frac{1}{j_2} \left\{ -(m_{13} - j_1 m_{12} + j_1 j_2)(m_{33} - j_1 m_{12} - j_1 j_2)(j_2 m_{23} - m_{12}) \right. 
\]

\[
\left. \left( \frac{j_2 m_{11} - m_{12}}{m_{22} - m_{12} - j_2} \right) \right\}^{1/2} \left| m_{12} - j_2 \right| +
\]

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\[+ \frac{1}{j_2} \left\{ - (m_{13} - j_1 m_{22} + 2j_1 j_2)(m_{33} - j_1 m_{22})(j_2 m_{23} + j_2 - m_{22}) \right. \]
\[\frac{(j_2 m_{11} + j_2 - m_{22})}{(m_{12} - m_{22} + 2j_2)(m_{12} - m_{22} + j_2)} \left\}^{1/2} |m_{22} - j_2\right.\]
\[A_{10}|m\rangle = \frac{1}{j_2} \left\{ - (m_{13} - j_1 m_{33} - j_1 m_{12} - 2j_1 j_2)(j_2 m_{23} - j_2 - m_{12}) \right. \]
\[\frac{(j_2 m_{11} - j_2 - m_{12})}{(m_{22} - m_{12} - j_2)(m_{22} - m_{12} - 2j_2)} \left\}^{1/2} |m_{12} + j_2\right.\]
\[+ \left\{ - (m_{13} - j_1 m_{22} + j_1 j_2)(m_{33} - j_1 m_{22} - j_1 j_2)(j_2 m_{23} - m_{22}) \right. \]
\[\frac{(j_2 m_{11} - m_{22})}{(m_{12} - m_{22} + j_2)(m_{12} - m_{22})} \left\}^{1/2} |m_{22} + j_2\right.\]
\[A_{02}|m\rangle = \left\{ - (m_{13} - j_1 m_{12} + j_1 j_2)(m_{33} - j_1 m_{12} - j_1 j_2)(j_2 m_{23} - m_{12}) \right. \]
\[\frac{(m_{22} - j_2 m_{11})}{(m_{22} - m_{12})(m_{22} - m_{12} - j_2)} \left\}^{1/2} |m_{12} - j_2\right.\]
\[\frac{(m_{12} - j_2 m_{11} + j_2)}{(m_{12} - m_{22} + j_2)(m_{12} - m_{22} + j_2)} \left\}^{1/2} |m_{22} - j_2\right.\]
\[A_{20}|m\rangle = \left\{ (m_{13} - j_1 m_{12})(m_{33} - j_1 m_{22} - 2j_1 j_2)(j_2 m_{23} - j_2 - m_{12}) \right. \]
\[\frac{(m_{22} - j_2 m_{11} - j_2)}{(m_{22} - m_{12} - j_2)(m_{22} - m_{12} - 2j_2)} \left\}^{1/2} |m_{12} + j_2\right.\]
\[\frac{(m_{12} - j_2 m_{11} + j_2)}{(m_{12} - m_{22} + j_2)(m_{12} - m_{22})} \left\}^{1/2} |m_{22} + j_2\right.\]

where \(|m_{12} \pm j_2\rangle\) is the scheme (25), in which component \(m_{12}\) is substituted for \(m_{12} \pm j_2\) and so on. Generators \(A_{11}, A_{22}, A_1, A_{21}\), making subalgebra \(u(2; j_2)\), are described by (7), where each index of generators must be increased by unit and parameter \(j_1\) has to be substituted for parameter \(j_2\).

Generators (26) satisfy the commutation relations of algebra \(u(3; j)\):

\[\left[ A_{kr}, A_{pq} \right] = \frac{J_{kr} J_{pq}}{J_{kq}} \delta_{pr} A_{kq} - \frac{J_{kr} J_{pq}}{J_{pr}} \delta_{kq} A_{pr}, \]  
\[J_{kr} = \prod_{l=1+\min(k,r)}^{\max(k,r)} i_l, \quad k, r, p, q = 0, 1, 2, \]

coinciding with Cartan-Weyl commutation relations in [2] for \(A_{kk} = H_k, A_{kr} = E_{k-r} \).
Unitary algebra $u(3)$ has three Casimir operators, which under transition to algebra $u(3; j)$ are transformed as follows [2]:

$$C_1(j) = C_1^1(-), \quad C_2(j) = j_1^2j_2^2C_2^2(-), \quad C_3(j) = j_1^2j_2^2C_3^2(-).$$

Spectrum of Casimir operators in this case is as follows

$$C_1(j) = \frac{m_{13} + m_{33}}{j_1j_2} + m_{23},$$

$$C_2(j) = m_{13}^2 + m_{33}^2 + j_1^2j_2^2m_{23}^2 + 2j_1j_2(m_{23} - m_{33}),$$

$$C_3(j) = \frac{1}{j_1j_2}(m_{13}^3 + m_{33}^3) + 2(2m_{13}^2 - m_{33}^2) - m_{13}m_{33} +$$

$$+ j_1^2j_2^2(m_{23}^2 + 2m_{23}^2 - 2m_{23}) + j_1j_2[2(m_{23} - m_{33}) - m_{23}(m_{13} + m_{33})].$$

This naturally brings up the question: for what considerations has been chosen transformation rule (24) for components of Gel’fand-Zetlin schemes or rule (6) in the case of algebra $u(2; j)$? We choose it in order to make spectrum of Casimir operators of second order different from zero and not involving undeterminate expressions for dual values of parameters $j$. Because $C_2(j) = j_1^2j_2^2C_2^2(-)$ and components $m_{13}, m_{23}, m_{33}$ enter $C_2^2$ quadratically, this requirement gives (24). However, variant (24) (we call it basic) is not unique. Two other variants are possible as well: $m_{13} = m_{13}^*, m_{23} = j_1j_2m_{23}^*, m_{33} = j_1j_2m_{33}^*$ or $m_{13} = j_1j_2m_{13}^*, m_{23} = j_1j_2m_{23}^*, m_{33} = m_{33}^*$, which turn initial irreducible representation of algebra $u(3)$ into representations of algebra $u(3; j)$ with other (in comparison with basic invariant (28)) values of Casimir operators. For example, $C_2(j) = m_{13}^2 + m_{33}^2 + j_1^2j_2^2m_{13}(m_{13} + 2) - 2j_1j_2m_{23}$ and $C_2'(j) = m_{13}^2 + m_{23}^2 + j_1^2j_2^2m_{33}(m_{33} - 2) + 2j_1j_2m_{13}$. The consideration of these variants of transition for irreducible representations is quite similar to basic variant, and we skip the corresponding relations.

It will be shown further that basic transformations (24), as two other variants, give under contractions general, non-degenerate representations of contracted algebras, all Casimir operators, which are independent and have non-zero spectrum.

For interpretation of formal inequalities (25) let us consider the action of rising generator $A_{10}$ on the vector of the “major weight” $\varphi_{M_{uw}}$, described by scheme (??), for $m_{11} = m_{12} = m_{13}$, $m_{22} = m_{23}$, and the action of lowering generator $A_{01}$ on the vector of the “minor weight” $\varphi_{uw}$, described by scheme (25) for $m_{11} = m_{22} = m_{33}$, $m_{12} = m_{23}$. Let us write out explicitly only those factors, which vanish for $j_1 = j_2 = 1$. Then

$$A_{10}\varphi_{M_{uw}} = \frac{1}{j_2}(m_{13}(1 - j_1)A)^{1/2}|m_{13} + j_2) + \frac{1}{j_2}(m_{23}(j_2 - 1)B)^{1/2}|m_{23} + j_2),$$

$$A_{01}\varphi_{uw} = \frac{1}{j_2}(m_{23}(j_2 - 1)C)^{1/2}|m_{23} - j_2) + \frac{1}{j_2}(m_{33}(1 - j_1)D)^{1/2}|m_{33} - j_2).$$

It can be seen from here that for $j_1 \neq 1, j_2 = 1$ $A_{10}\varphi_{M_{uw}} = \{m_{13}(1 - j_1)A\}^{1/2}|m_{13} + 1) \neq 0$, which means the absence of the bound from above on $m_{13}$; $A_{01}\varphi_{uw} = \{m_{33}(1 - j_1)D\}^{1/2}|m_{33} - 1) \neq 0$, that means the absence of the bound from below on $m_{22}$, i.e. components of scheme (26) satisfy inequalities $m_{12} \geq m_{23} \geq m_{22}$. For $j_1 = 1, j_2 \neq 1$ we obtain from (29) $A_{10}\varphi_{M_{uw}} = \frac{1}{j_2}(m_{23}(j_2 - 1)B)^{1/2}|m_{23} + j_2) \neq 0$, that means the absence of the bound from above on $m_{22}$, and $A_{01}\varphi_{uw} = \frac{1}{j_2}(m_{23}(j_2 - 1)C)^{1/2}|m_{23} - j_2) \neq 0$, which means the absence of the bound from below on $m_{12}$, i.e. components of scheme (25) satisfy inequalities $m_{13} \geq m_{12}$, $m_{22} \geq m_{33}$.
$-\infty < m_{11} < \infty$. At last, we find from (29) for $j_1 \neq 1$, $j_2 \neq 1$ that there are no restrictions for components $m_{12}$, $m_{22}$, $m_{11}$. In all cases inequality $m_{13} \geq m_{33}$ remains valid.

The same inequalities for components of Gel’fand-Zetlin scheme can be derived from formal inequalities (25), if one interprets them for $j_1, j_2 \neq 1$ according to following rules: inequality $\frac{m}{j_2} \geq m_1$ means the absence of the bounds from above on $m_1$; inequality $\frac{m_1}{j_2} \geq \frac{m}{j_2}$ means the absence of the bounds from below on $m_1$; inequality $\frac{m_1}{j_2} \geq \frac{m}{j_2}$ is equivalent to $\frac{m}{j_1} \geq m_1$, i.e. common parameters in both parts of inequality can be cancelled out. The same rules are valid for algebras of higher dimensions as well.

Formulas for irreducible representations of algebra $u(3)$ can be obtained from formulas of this paragraph for $j_1 = j_2 = 1$. The requirement of unitarity for representations of algebra $u(3)$ leads to the following relations for operators (26): $A_{kk} = \bar{A}_{kk}$ ($k = 0, 1, 2$), $A_{rp} = \bar{A}_{rp}$ ($r, p = 0, 1, 2$). Here the bar means complex conjugation.

2.2 Contraction over the first parameter

The structure of contracted unitary algebra, described in [2], is as follows: $u(3; \xi_1, j_2) = T_1(u(1) \oplus u(2; j_2))$, where $T_1 = \{A_0, A_{10}, A_{02}, A_{20}\}$; $u(2; j_2) = \{A_{11}, A_{22}, A_{12}, A_{21}\}$; $u(1) = \{A_{00}\}$. The relations (26) give for $j_1 = 1$:

$$A_{00}|m\rangle = \left(\frac{m_{13} + m_{33}}{\xi_1 j_2} + m_{23} - \frac{m_{12} + m_{22}}{j_2}\right)|m\rangle,$$

$$A_{01}|m\rangle = \frac{1}{\sqrt{j_2}} \{-m_{13}m_{33}\} \left(\left\{\frac{(j_2m_{23} - j_2)(j_2m_{11} - j_2)}{(\xi_1 j_2)(m_{22} - m_{12} - j_2)} \right\}^{1/2} |m_{12} - j_2\rangle + \left\{\frac{(j_2m_{23} + j_2 - m_{22})(j_2m_{11} + j_2 - m_{22})}{(m_{12} - m_{22} + 2j_2)(m_{12} - m_{22} + j_2)} \right\}^{1/2} |m_{22} - j_2\rangle\right),$$

$$A_{10}|m\rangle = \frac{1}{\sqrt{j_2}} \{-m_{13}m_{33}\} \left(\left\{\frac{(j_2m_{23} - j_2 - m_{12})(j_2m_{11} - j_2 - m_{12})}{(m_{22} - m_{12} - j_2)(m_{22} - m_{12} - 2j_2)} \right\}^{1/2} |m_{12} + j_2\rangle + \left\{\frac{(j_2m_{13} - m_{22})(j_2m_{11} - m_{22})}{(m_{12} - m_{22} + j_2)(m_{12} - m_{22} + j_2)} \right\}^{1/2} |m_{22} + j_2\rangle\right),$$

$$A_{02}|m\rangle = \sqrt{-m_{13}m_{33}} \left(\left\{\frac{(j_2m_{23} - j_2)(m_{22} - j_2m_{11})}{(m_{22} - m_{12} - j_2)} \right\}^{1/2} |m_{12} - j_2\rangle + \left\{\frac{(j_2m_{23} + j_2 - m_{22})(m_{12} - j_2m_{11} + j_2)}{(m_{12} - m_{22} + 2j_2)(m_{12} - m_{22} + j_2)} \right\}^{1/2} |m_{22} - j_2\rangle \right),$$

$$A_{20}|m\rangle = \sqrt{-m_{13}m_{33}} \left(\left\{\frac{(j_2m_{23} - j_2 - m_{12})(m_{22} - j_2m_{11} - j_2)}{(m_{22} - m_{12} - j_2)(m_{22} - m_{12} - 2j_2)} \right\}^{1/2} |m_{12} - j_2\rangle + \left\{\frac{(j_2m_{23} - m_{22})(m_{12} - j_2m_{11})}{(m_{12} - m_{22} + j_2)(m_{12} - m_{22} + j_2)} \right\}^{1/2} |m_{22} - j_2\rangle \right).$$

(30)

Here dual parts, arising in the expressions for generators, are omitted, and only real parts are written.

Algebra $u(3; \xi_1, 1)$ is inhomogeneous algebra $iu(2)$ in Chakrabarti’s notations [7]. The requirement of determinacy and unitarity of generator $A_{00}$ gives $(m_{13} + m_{33})/\xi_1 = \xi \in R$, i.e.

$$m_{13} = k + \xi_1 \xi/2, m_{33} = -k + \xi_1 \xi/2, \quad \xi, k \in R, \quad k \geq 0.$$

(31)
Real-valuedness of \( k \) follows from unitary relations for \( A_{01}, A_{10} \), and its positiveness – from inequality \( m_{13} \geq m_{33} \), considered for real parts. Taking into account (31), we get \( \sqrt{-m_{13}m_{33}} = k \), and the expressions (30) for \( j_2 = 1 \) coincide with corresponding Chakrabarti’s formulas [7] for \( iu(2) \). The integer components of scheme \( |\tilde{m}\rangle \) are interrelated via inequalities \( m_{12} \geq m_{23} \geq m_{22} \), \( m_{12} \geq m_{11} \geq m_{22} \), ensued from (25) for \( j_1 = \iota_1 \). The scheme \( |\tilde{m}\rangle \) can be obtained from scheme (25) for \( m_{13} = k, m_{33} = -k \). Spectrum of Casimir operators in a given irreducible representation of algebra \( u(3; \iota_1, 1) \) can be found from (28):

\[
C_1(\iota_1, 1) = \xi + m_{23}, \quad C_2(\iota_1, 1) = 2k^2, \quad C_3(\iota_1, 1) = 3k^2(\xi + 1).
\]  

(32)

Algebra \( su(3; \iota_1, 1) \) differs from algebra \( u(3; \iota_1, 1) \) in that diagonal operators satisfy the relation \( A_{00} + A_{11} + A_{22} = 0 \). Acting on scheme \( |\tilde{m}\rangle \), we get \( \xi + m_{23}|\tilde{m}\rangle = 0 \), from which it follows \( \xi = -m_{23} \). Substituting \( \xi \) in (32), we find spectrum of Casimir operators

\[
C_1(\iota_1, 1) = 0, \quad C_2(\iota_1, 1) = 2k^2, \quad C_3(\iota_1, 1) = 3k^2(1 - m_{23})
\]  

(33)

of the irreducible representation of algebra \( su(3; \iota_1, 1) = T_4u(2) \), generators of which are described by (31) for \( j_2 = 1 \), where it is necessary to put

\[
m_{13} = k - \iota_1m_{23}/2, \quad m_{33} = -k - \iota_1m_{23}/2, \quad k \geq 0, \quad m_{23} \in \mathbb{Z}.
\]  

(34)

Here \( Z \) is a set of integers.

2.3 Contraction over the second parameter

The structure of contracted algebra is described in [2] and is as follows: \( u(3; j_1, \iota_2) = T_4(u(2; j_1) \oplus u(1)) \), where \( T_3 = \{ A_{12}, A_{21}, A_{02}, A_{20} \}; u(2; j_1) = \{ A_{00}, A_{11}, A_{01}, A_{10} \}; u(1) = \{ A_{22} \} \). After substitution of \( j_2 = \iota_2 \) in (26), expressions \( |m_{12} \pm \iota_2\rangle \) can occur, with which we proceed according to general rules (see [6]) of treating functions of dual variable, i.e. expand into series

\[
|m_{12} \pm \iota_2\rangle = |m\rangle \pm \iota_2 |m\rangle', |m\rangle'_{12} \equiv \frac{\partial}{\partial m_{12}} |m\rangle,
\]

\[
|m_{22} \pm \iota_2\rangle = |m\rangle \pm \iota_2 |m\rangle', |m\rangle'_{22} \equiv \frac{\partial}{\partial m_{22}} |m\rangle.
\]  

(35)

Taking this remark into account, (26) give for \( j_2 = \iota_2 \) the following expressions for generators:

\[
A_{00}|m\rangle = \left( m_{23} + \frac{m_{13} + m_{33}}{\iota_2} - m_{12} + m_{22}\right)|m\rangle,
\]

\[
A_{11}|m\rangle = \left( \frac{m_{12} + m_{22}}{\iota_2} - m_{11}\right)|m\rangle, \quad A_{22}|m\rangle = m_{11} |m\rangle,
\]

\[
A_{12}|m\rangle = \sqrt{-m_{12}m_{22}}|m_{11} - 1\rangle, \quad A_{21}|m\rangle = \sqrt{-m_{12}m_{22}}|m_{11} + 1\rangle,
\]

\[
A_{01}|m\rangle = \frac{1}{m_{12} - m_{22}} \left( \frac{1}{\iota_2} (m_{12}a_{12} + m_{22}a_{22}) |m\rangle + \left[ j_1m_{12}(m_{13} - m_{33}) - a_{12}^2 (m_{11} + m_{23} + \frac{m_{12}}{m_{12} - m_{22}}) \right]|m\rangle - \frac{1}{2a_{12}} \left[ 2j_1m_{22}(m_{33} - j_1m_{22}) + a_{22}^2 (m_{11} + m_{23} + \frac{2m_{12} + m_{22}}{m_{12} - m_{22}}) \right]|m\rangle - \right.
\]

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analytically continued algebras, corresponding to such schemes are understood to be generalized values. In this case basis vectors in infinite-dimensional space of representation for contracted or analytical continuations a part of components of schemes merate normalized to unit basis vectors in finite-dimensional space of representation. Under Hermiticity brings to components of schemes consider in denominators of (27) \( m \)

where only real parts are written. To obtain correct expressions for \( A_0 \), \( A_{10} \), it is necessary to consider in denominators of (27) \( m_{12} - m_{22} \).

The requirement of determinacy of generators \( A_{00}, A_{11} \) together with condition of their Hermiticity brings to components of schemes

\[
m_{13} = k + \iota_2 j_1 \xi/2, \quad m_{33} = -k + \iota_2 j_1 \xi/2, \quad k \geq 0, \xi \in R, \\
m_{12} = r + \iota_2 \zeta/2, \quad m_{22} = -r + \iota_2 \zeta/2, \quad r \geq 0, \quad \zeta \in R. \tag{37}
\]

Substituting components (37) in (25), we get

\[
|m\rangle = |\tilde{m}\rangle + \iota_2 j_1 \xi/2(|\tilde{m}\rangle'_{13} + |\tilde{m}\rangle'_{33}) + \iota_2 \zeta/2(|\tilde{m}\rangle'_{12} + |\tilde{m}\rangle'_{22}),
\]

\[
|\tilde{m}\rangle = \begin{pmatrix} k & m_{23} & -k \\ r & m_{11} & -r \\ m_{12} & m_{22} & 0 \end{pmatrix}, \quad m_{11}, m_{23} \in Z. \tag{38}
\]

For classical unitary algebras, Gel’fand-Zetlin schemes \( |m\rangle \) with integer components enumerate normalized to unit basis vectors in finite-dimensional space of representation. Under contraction and analytical continuations a part of components of schemes \( |m\rangle \) takes continuous values. In this case basis vectors in infinite-dimensional space of representation for contracted or analytically continued algebras, corresponding to such schemes are understood to be generalized functions, orthogonal as before, but normalized to delta-function. In particular, for \( |\tilde{m}\rangle \) we get

\[
\langle \tilde{m}' | \tilde{m}\rangle = \delta^2(k' - k)\delta^2(r' - r)\delta_{m_{23}', m_{23}}\delta_{m_{11}', m_{11}}, \tag{39}
\]

where the squared delta-functions occur due to the fact that \( r \) and \( k \) twice enter the components of scheme (for details see [12]–[14] by Celeghini in the case of contractions and [15], [16] in the case of analytical continuations).
Substituting (37), (38) in (36), we obtain generators of irreducible representation of algebra $u(3; j_1, \iota_2)$:

$$A_{00}|\tilde{m}\rangle = (m_{23} + \xi - \zeta)|\tilde{m}\rangle, \quad A_{11}|\tilde{m}\rangle = (\zeta - m_{11})|\tilde{m}\rangle, \quad A_{22}|\tilde{m}\rangle = m_{11}|\tilde{m}\rangle,$$

$$A_{12}|\tilde{m}\rangle = r|m_{11} - 1\rangle, \quad A_{11}|\tilde{m}\rangle = r|m_{11} + 1\rangle,$$

$$A_{02}|\tilde{m}\rangle = \sqrt{k^2 - J_1^2 r^2}|m_{11} - 1\rangle, \quad A_{20}|\tilde{m}\rangle = \sqrt{k^2 - J_1^2 r^2}|m_{11} + 1\rangle,$$

$$A_{01}|\tilde{m}\rangle = \frac{1}{2r}\sqrt{k^2 - J_1^2 r^2}\left\{\left(\zeta - m_{11} - m_{23} - \frac{1}{2}\right)|\tilde{m}\rangle + \right.$$

$$+ \left.J_1^2 r^2 \frac{\xi - \zeta + 1}{k^2 - J_1^2 r^2} |\tilde{m}\rangle - r(|\tilde{m}\rangle_{12} - |\tilde{m}\rangle_{22})\right\},$$

$$A_{10}|\tilde{m}\rangle = \frac{1}{2r}\sqrt{k^2 - J_1^2 r^2}\left\{\left(\zeta - m_{11} - m_{23} + \frac{1}{2}\right)|\tilde{m}\rangle + \right.$$

$$+ \left.J_1^2 r^2 \frac{\xi - \zeta - 1}{k^2 - J_1^2 r^2} |\tilde{m}\rangle + r(|\tilde{m}\rangle_{12} - |\tilde{m}\rangle_{22})\right\}. \quad (40)$$

The relation of Hermiticity for operators $A_{02}, A_{20}$ gives $k^2 - J_1^2 r^2 \geq 0$, which for $j_1 = 1$ imposes restriction $k \geq r$. The action of operators on the derived schemes can be found, using (36), by application of operators to both sides of equation $|m\rangle_{12}' = \frac{1}{2r}(|m_{12} + \iota_2) - |m_{12} - \iota_2\rangle$.

The eigenvalues of Casimir operators for representation (40) can be obtained by substituting of components ($\xi, \zeta$).

They are all different from zero and independent, as it must be for non-degenerate irreducible representations of algebra $u(3; j_1, \iota_2)$. Let us notice that spectrum (32) coincides with spectrum (32) of Casimir operators for algebra $u(3; \iota_1, j_2)$.

For the sake of convenience of applications (interpretation) we have fixed indices of generators $A_{ps}$, for this reason $u(3; 1, \iota_2)$ and $u(3; \iota_1, 1)$ turned out in our case to be different algebras. Rejecting this agreement it is easy to prove that these algebras are isomorphic. Representation (30), (31) is realized in discrete basis, generated by the chain of subalgebras $u(3; \iota_1, 1) \supset u(2; 1) \supset u(1)$ and described by schemes:

$$\begin{pmatrix} k & m_{23} & -k \\ m_{12} & m_{22} & m_{11} \end{pmatrix}, \quad m_{12} \geq m_{23}, \quad m_{11} \geq m_{22}, \quad m_{12} \geq m_{11}, \quad k \geq 0, \quad k \geq 0, \quad m_{23} \in Z, \quad m_{23} \in Z, \quad m_{23} \in Z, \quad m_{23} \in Z, \quad m_{23} \in Z. \quad (41)$$

whereas representation (40) is realized in continuous basis, generated by expansion $u(3; 1, \iota_2) \supset u(2; \iota_2) \supset u(1)$ and described by schemes

$$\begin{pmatrix} k & m_{23} & -k \\ r & m_{11} & -r \end{pmatrix}, \quad m_{23}, \quad m_{11} \in Z, \quad m_{23}, \quad m_{11} \in Z, \quad k \geq r \geq 0, \quad k \geq r \geq 0. \quad (42)$$

where besides $A_{00}, A_{11}, A_{22}$ operator $A_{01} + A_{10}$ is also diagonal in this basis.

Thus, contractions over different parameters, leading to isomorphic algebras, give description of the same irreducible representation of contracted algebra in different bases, generated by canonical chains of subalgebras.
2.4 Two-dimensional contraction

The structure of algebra \(u(3; \iota)\) is given in [2]. It is as follows: \(u(3; \iota) = T_0(\{A_{00}\} \oplus \{A_{11}\} \oplus \{A_{22}\})\), where nilpotent subalgebra \(T_0\) is spanned over generators \(A_{ps}, p, s = 1, 2\). The explicit form of generators of irreducible representations of algebra can be obtained, putting either \(j_1 = \iota_1, j_2 = \iota_2\) in (26), or \(j_2 = \iota_2\) in (30), or from (36) for \(j_1 = \iota_1\). All three approaches lead to the same result:

\[
A_{00}|m\rangle = \left( m_{23} + \frac{m_{13} + m_{33}}{\iota_{12}} - \frac{m_{12} + m_{22}}{\iota_{2}} \right) |m\rangle,
A_{22}|m\rangle = m_{11} |m\rangle,
A_{11}|m\rangle = \left( \frac{m_{12} + m_{22}}{\iota_{2}} - m_{11} \right) |m\rangle, \quad A_{12}|m\rangle = \alpha |m_{11} - 1\rangle,
A_{21}|m\rangle = \alpha |m_{11} + 1\rangle,
A_{02}|m\rangle = \frac{2\alpha \beta}{m_{12} - m_{22}} |m_{11} - 1\rangle, \quad A_{20}|m\rangle = \frac{2\alpha \beta}{m_{12} - m_{22}} |m_{11} + 1\rangle,
A_{01}|m\rangle = \frac{\beta}{m_{12} - m_{22}} \left\{ \frac{m_{12} + m_{22}}{\iota_{2}} |m\rangle - \left( m_{11} + m_{23} + \frac{3m_{12} + m_{22}}{2(m_{12} - m_{22})} \right) |m\rangle - m_{12} |m_{12} - m_{22}\rangle_{22} \right\} ,
A_{10}|m\rangle = \frac{\beta}{m_{12} - m_{22}} \left\{ \frac{m_{12} + m_{22}}{\iota_{2}} |m\rangle - \left( m_{11} + m_{23} + \frac{m_{12} + 3m_{22}}{2(m_{12} - m_{22})} \right) |m\rangle + m_{12} |m_{12} + m_{22}\rangle_{22} \right\} ,
\alpha = \sqrt{-m_{12} m_{22}}, \quad \beta = \sqrt{-m_{13} m_{33}}.
\]

The requirement of determinacy of operators \(A_{00}, A_{11}\) and condition of Hermiticity, give for the components of scheme \(|m\rangle\):

\[
m_{13} = k + \iota_1 \iota_2 \xi/2, \quad m_{33} = -k + \iota_1 \iota_2 \xi/2, \quad k \geq 0, \quad \xi \in R, \quad m_{12} = r + \iota_2 \zeta/2, \quad m_{22} = -r + \iota_2 \zeta/2, \quad r \geq 0, \quad \zeta \in R.
\]

The substitution of these expressions in (43) leads to representation operators

\[
A_{00}|\tilde{m}\rangle = (m_{23} + \xi - \zeta)|\tilde{m}\rangle, \quad A_{11}|\tilde{m}\rangle = (\zeta - m_{11})|\tilde{m}\rangle, \quad A_{22}|\tilde{m}\rangle = m_{11}|\tilde{m}\rangle, \quad A_{12}|\tilde{m}\rangle = r|m_{11} - 1\rangle, \quad A_{21}|\tilde{m}\rangle = r|m_{11} + 1\rangle,
A_{02}|\tilde{m}\rangle = k|m_{11} - 1\rangle, \quad A_{20}|\tilde{m}\rangle = k|m_{11} + 1\rangle,
A_{01}|\tilde{m}\rangle = \frac{k}{2r} \left( \zeta - m_{11} - m_{23} - \frac{1}{2} \right) |\tilde{m}\rangle - \frac{k}{2} (|\tilde{m}\rangle_{12} - |\tilde{m}\rangle_{22}),
A_{10}|\tilde{m}\rangle = \frac{k}{2r} \left( \zeta - m_{11} - m_{23} + \frac{1}{2} \right) |\tilde{m}\rangle + \frac{k}{2} (|\tilde{m}\rangle_{12} - |\tilde{m}\rangle_{22}),
\]

where \(|\tilde{m}\rangle\) means the scheme

\[
|\tilde{m}\rangle = \begin{pmatrix} k & m_{23} & -k \\ r & m_{11} & -r \\ m_{11} & -r & k \end{pmatrix}, \quad m_{11}, m_{23} \in Z, \quad k \geq 0, \quad r \geq 0.
\]

It is worth of mentioning that operator \(A_{01} + A_{10}\) is diagonal in basis \(|\tilde{m}\rangle\), and spectrum of Casimir operators for irreducible representations (45) of algebra \(u(3; \iota)\) is given by the same formulas (32), (??) as in the case of algebras \(u(3; \iota_1, \iota_2), u(3; \iota_1, \iota_2)\).
3 Representations of unitary algebras \( u(n; j) \)

3.1 Operators of representation

Standard notations of Gel’fand-Zetlin [11] correspond to diminishing chain of subalgebras \( u(n) \supseteq u(n-1) \supseteq \ldots \supseteq u(2) \supseteq u(1) \), where \( u(n) = \{E_{kr}, k, r = 1, 2, \ldots, n\} \), \( u(n-1) = \{E_{kr}, k, r = 1, 2, \ldots, n-1\} \), \( u(2) = \{E_{kr}, k, r = 1, 2\} \), \( u(1) = \{E_{11}\} \). We shall use now another imbedding of subalgebra into algebra, which leads to the chain of subalgebras \( u(n; j_1, j_2, \ldots, j_{n-1}) \supseteq u(n-1; j_2, \ldots, j_{n-1}) \supseteq \ldots \supseteq u(2; j_{n-1}) \supseteq u(1) \), where \( u(n; j_1, \ldots, j_{n-1}) = \{A_{sp}, s, p = 0, 1, \ldots, n-1\} \), \( u(n-1; j_2, \ldots, j_{n-1}) = \{A_{sp}, s, p = 1, 2, \ldots, n-1\} \), \( u(2; j_{n-1}) = \{A_{sp}, s, p = n-2, n-1\} \), \( u(1) = \{A_{n-1,n-1}\} \). To pass from standard notations to ours, it is necessary to change index \( k \) of generator for index \( n-k \) and to leave unchanged the numbering of components in Gel’fand-Zetlin schemes.

To determine representations of algebra \( u(n) \), it is sufficient to give the action of generators \( E_{kk}, E_{k,k+1}, E_{k+1,k} \) and to find the rest generators from commutators. In our notations it is sufficient to know generators \( A_{n-k,n-k}, A_{n-k,n-k-1}, A_{n-k-1,n-k}, A_{n-k-1,n-k-1}, \) which are transformed under transition from \( u(n) \) to \( u(n;j) \) as follows:

\[
A_{n-k,n-k-1} = j_{n-k} A_{n-k,n-k-1}^* (-\rightarrow), \tag{47}
\]

\[
A_{n-k-1,n-k} = j_{n-k} A_{n-k-1,n-k}^* (-\rightarrow), \quad k = 1, 2, \ldots, n-1, \]

where \( j_{n-k} \) for dual value plays the role of tending to zero parameter in Wigner-Inemu contraction [17]: \( A^* (-\rightarrow) \) is singularly transformed generator.

To give of a singular transformation is equivalent as to give the transformation rule for components of Gel’fand-Zetlin scheme

\[
|m^*\rangle = \begin{pmatrix}
m_{1n}^* & m_{1,n-1}^* & m_{2n}^* & \cdots & m_{n-1,n}^* & m_{nn}^*
m_{1,n-1}^* & m_{2,n-1}^* & \cdots & \cdots & \cdots & \cdots
m_{12}^* & m_{22}^* & \cdots & \cdots & \cdots & \cdots
m_{11}^*
\end{pmatrix},
\]

\[
m_{p,k}^* \geq m_{p,k-1}^* \geq m_{p+1,k}^*, \quad k = 2, 3, \ldots, n, \quad p = 1, 2, \ldots, n-1,
\]

\[
m_{1n}^* \geq m_{2n}^* \geq \cdots \geq m_{nn}^*
\]

under the transition from \( u(n) \) to \( u(n;j) \). Defining this transformation by

\[
m_{1k} = m_{1k}^* J_k, \quad m_{kk} = m_{kk}^* J_k, \quad J_k = \prod_{l=n-k+1}^{n-1} j_l,
\]

\[
m_{pk} = m_{pk}^*, \quad p = 2, 3, \ldots, n-1, \quad k = 2, 3, \ldots, n,
\]

we obtain the scheme \( |m\rangle \), which components \( m_{pk} \) are integers, and components \( m_{1k}, m_{kk} \) can be complex or dual numbers. They satisfy inequalities

\[
m_{pk} \geq m_{p,k-1} \geq m_{p+1,k}, \quad k = 2, 3, \ldots, n, \quad p = 2, 3, \ldots, n-2,
\]

\[
\frac{m_{1k}}{J_k} \geq \frac{m_{1,k-1}}{J_{k-1}} \geq m_{2k}, \quad m_{k-1,k} \geq \frac{m_{k-1,k-1}}{J_{k-1}} \geq \frac{m_{kk}}{J_k}, \tag{50}
\]

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\[
\frac{m_{1n}}{J_n} \geq m_{2n} \geq m_{3n} \geq \ldots \geq m_{n-1,n} \geq \frac{m_{nn}}{J_n},
\]

which for dual and imaginary values of parameters \( j \) are interpreted according to the rules, described in §2.1.

Substituting (49) in known expressions for generators of algebra and taking into account (47), we find operators of representation of algebra \( u(n;j) \):

\[
A_{n-k, n-k} | m \rangle = \left( \frac{m_{1k} + m_{kk}}{J_k} - \frac{m_{1k-1} + m_{k-1,k-1}}{J_{k-1}} + m_{k-1,k} + \right.
\]
\[
\left. + \sum_{s=2}^{k-2} (m_{sk} - m_{s,k-1}) \right) | m \rangle, \quad k = 1, 2, \ldots, n,
\]

\[
A_{n-k, n-k} | m \rangle = \frac{1}{J_k} \left[ \tilde{a}^1_k(m) | m_{1k} - J_k \rangle + \tilde{a}^k_k(m) | m_{kk} - J_k \rangle + \right.
\]
\[
\left. + j_{n-k+1} \sum_{s=2}^{k-1} \tilde{a}^s_k(m) | m_{sk} - 1 \rangle \right),
\]

\[
A_{n-k, n-k} | m \rangle = \frac{1}{J_k} \left[ \tilde{b}^1_k(m) | m_{1k} + J_k \rangle + \tilde{b}^k_k(m) | m_{kk} + J_k \rangle + \right.
\]
\[
\left. + j_{n-k+1} \sum_{s=2}^{k-1} \tilde{b}^s_k(m) | m_{sk} + 1 \rangle, \quad k = 1, 2, \ldots, n - 1,
\]

where

\[
\tilde{a}^1_k(m) = \left\{ \prod_{p=2}^{k} (J_k l_{p,k+1} - l_{1k} + J_k) \left( \prod_{p=2}^{k-2} (J_k l_{p,k-1} - l_{1k}) \prod_{p=2}^{k-1} (J_k l_{pk} - l_{1k} + J_k)^{-1} \right)^{-1} \right\}^{1/2}
\]

\[
\tilde{a}^k_k(m) = \left\{ \prod_{p=2}^{k} (l_{p,k+1} - l_{sk} + 1) \left( \prod_{p=2}^{k-2} (l_{p,k-1} - l_{sk}) \prod_{p=2, p \neq s}^{k-1} (l_{pk} - l_{sk} + 1)^{-1} \right)^{-1} \right\}^{1/2}
\]

\[
\tilde{a}^s_k(m) = \left\{ \prod_{p=2}^{k} (l_{p,k+1} - l_{sk}) \left( \prod_{p=2}^{k-2} (l_{p,k-1} - l_{sk}) \prod_{p=2, p \neq s}^{k-1} (l_{pk} - l_{sk} + 1)^{-1} \right)^{-1} \right\}^{1/2}, \quad 1 < s < k,
\]

\[
\tilde{b}^1_k(m) = \left\{ \prod_{p=2}^{k} (J_k l_{p,k+1} - l_{1k} + J_k) \prod_{p=2}^{k-2} (J_k l_{p,k-1} - l_{1k} - J_k) \prod_{p=2}^{k-1} (J_k l_{pk} - l_{1k} - J_k)^{-1} \right\}^{1/2}
\]

\[
\tilde{b}^k_k(m) = \left\{ \prod_{p=2}^{k} (l_{p,k+1} - l_{sk}) \left( \prod_{p=2}^{k-2} (l_{p,k-1} - l_{sk}) \prod_{p=2, p \neq s}^{k-1} (l_{pk} - l_{sk} + 1)^{-1} \right)^{-1} \right\}^{1/2}
\]

\[
\tilde{b}^s_k(m) = \left\{ \prod_{p=2}^{k} (l_{p,k+1} - l_{sk}) \left( \prod_{p=2}^{k-2} (l_{p,k-1} - l_{sk}) \prod_{p=2, p \neq s}^{k-1} (l_{pk} - l_{sk} + 1)^{-1} \right)^{-1} \right\}^{1/2}
\]
\[
\left\{ \frac{-(l_{1,k+1} - J_{k+1,l}) (l_{k+1,l} - J_{k+1,l}) (l_{1,k-1} - J_{k-1,l}) - J_{k-1,l}}{(l_{1,k} - J_{k,l})(l_{1,k} - J_{k,l})(l_{k,k} - J_{k,l})} \right\}, \quad 1 < s < k,
\]

The expression for \( \tilde{a}_k^s(m) \) can be derived from that for \( \tilde{a}_1^s(m) \) by changing \( l_{1,k} \) for \( l_{k,k} \) and \( l_{k,k} \) for \( l_{1,k} \). The same substitution turns \( \tilde{b}_1^s(m) \) into \( \tilde{b}_k^s(m) \). Components \( m \) are related with components \( l \) by equations

\[
l_{1,k} = m_{1,k} - J_k, \quad l_{k,k} = m_{k,k} - kJ_k, \quad l_{s,k} = m_{s,k} - s, \quad 1 < s < k.
\] (53)

As it can be shown by direct checking, operators (51) satisfy commutation relations (27) of algebra \( u(n; j) \). Therefore, they give a representation of algebra. Considering the action of rising operators \( A_{n-k,n-k-1} \) on vector of the “major weight” \( \varphi_{mw} \), described by scheme \( | m \rangle \) for maximal values of components, and the action of lowering operators \( A_{n-k-1,n-k} \) on vector of the “minor weight” \( \varphi_{mw} \), described by scheme \( | m \rangle \) for minimal values of components, as in \( \S 2.1 \), we find that for dual or imaginary values of all or some parameters \( j \) the space of representation is infinite-dimensional and does not contain subspaces invariant in respect to operators (51), because taking any basis vector and acting on it by operators \( A_{kr} \) required number of times, we obtain all basis vectors in the space of representation. Therefore, representation (51) is irreducible.

Though the initial representation of algebra \( u(n) \) is Hermitian, irreducible representation (51) of algebra \( u(n; j) \), in general, is not Hermitian. Therefore, if we want representation (51) to be Hermitian, it is necessary to require the fulfilment of relations \( A_{pp} = A_{pp} \) \( (p = 0, 1, \ldots, n-1) \), \( A_{kp} = A_{pk}^+ \), which for matrix elements of operators can be written as follows:

\[
\langle m | A_{pp} | m \rangle = \langle m | A_{pp}^\ast | m \rangle, \quad \langle n | A_{kp} | m \rangle = \langle m | A_{pk} | n \rangle,
\] (54)

where bar means complex conjugation.

### 3.2 Spectrum of Casimir operators

Components \( m_{1,m}^s \) of the upper row of scheme (48) (components of the highest weight) completely determine the irreducible representation of algebra \( u(n) \). A.M.Perelomov and V.S.Popov [3], A.N.Leznov, I.A.Malkin, V.I.Man’ko [5] found eigenvalues of Casimir operators, expressing them in terms of components of the major weight. For unitary algebra \( u(n) \) spectrum of Casimir operators can be written as follows:

\[
C_S^a(m^\ast) = Tr a^\ast d E,
\] (55)

where \( E \) is matrix of dimension \( n \), all elements of which are equal to unit, and matrix \( a^\ast \) is as follows

\[
a^\ast_{ps} = (m_{mn}^s + n - p) \delta_{ps} - \theta_{sp}, \quad s, p = 1, 2, \ldots, n.
\] (56)

Here \( \theta_{sp} = 1 \) for \( s < p \) and \( \theta_{sp} = 0 \) for \( s \geq p \).

Under transition from algebra \( u(n) \) to algebra \( u(n; j) \), \( j = (j_1, j_2, \ldots, j_{n-1}) \) the components of major weight are transformed according to (49), i.e. \( m_{1n} = Jm_{1n}^s, m_{nn} = Jm_{nn}^s, m_{sn} = m_{sn}^s \) \( (s = 2, 3, \ldots, n-1) \), \( J = \prod_{l=1}^{n-1} j_l \). Let us define matrix \( a(j) \) as

\[
a(j) = Ja^\ast(-),
\] (57)
where $a^*(\rightarrow)$ is matrix (56), in which the components $m^*_{pn}$ are substituted by their expressions in terms of $m_{pm}$, i.e. $a_{11}(\rightarrow) = n - 1 + m_{1n}J^{-1}$, $a_{nn}(\rightarrow) = m_{nn}J^{-1}$, and the rest matrix elements are given by (56). Then matrix $a(j)$ is as follows:

$$
a_{11}(j) = m_{1m} + J(n - 1), \quad a_{nn}(j) = m_{nn}, \quad (58)
$$

$$
a_{ps}(j) = J[(m_{pn} + n - p)\delta_{ps} - \theta_{sp}], \quad p, s = 2, 3, \ldots, n - 2.
$$

Casimir operators are transformed according to [2]: $C_{2k}(\rightarrow) = J^{2k}C^*_{2k}(\rightarrow)$, $C_{2k+1}(\rightarrow) = J^{2k}C_{2k+1}(\rightarrow)$. Their spectra are transformed in just the same way. Therefore, spectrum of Casimir operators for algebra $u(n;j)$ is as follows:

$$
C_{2k}(m) = J^{2k}Tr(a^*(\rightarrow))^{2k}E = Tr(Ja^*(\rightarrow))^{2k}E = Tr\eta(j)E, \quad (59)
$$

$$
C_{2k+1}(m) = J^{2k}Tr(a^*(\rightarrow))^{2k+1}E = Tr(a^*(\rightarrow)Ja^*(\rightarrow))^{2k}E = Tr\eta^*(\rightarrow)a^{2k}(j)E,
$$

where $2k$ and $2k + 1$ takes all integer values from 1 to $n$. In particular,

$$
C_1(m) = (m_{1n} + m_{nn})J^{-1} + \sum_{s=2}^{n-1} m_{sn}, \quad (60)
$$

$$
C_2(m) = m^2_{1n} + m^2_{nn} + J(n - 1)(m_{1n} - m_{nn}) + J^2\sum_{s=2}^{n-1} m_{sn}(m_{sn} + n + 1 - 2s)
$$

are eigenvalues of the first two Casimir operators of algebra $u(n;j)$ on the irreducible representation.

### 3.3 Possible variants of contractions of irreducible representations

For brevity, in this section we shall talk on contractions of irreducible representations, however, keeping in mind that the corresponding considerations are valid for imaginary values of parameters $j$ as well. Transformation (49) of components in Gel’fand-Zetlin scheme have been chosen in such a way that eigenvalues of Casimir operators of even order would differ from zero under contractions. However, variant (49) (we call it basic) is not unique. As it can be easily seen from (59), (60), the same goal can be achieved, transforming any two components of the upper row according to the rule $m = Jm^*$ and leaving unchanged the other components of this row.

What happens in this case with initial irreducible representation, say under contraction $j_1 = \iota_1, j_2 = \ldots = j_{n-1} = 1^?$ The transformation rule for generators remains unchanged: $A = (\prod_j J_j)A^*(\rightarrow)$, only expressions $A^*(\rightarrow)$ for singularly transformed operators of representation are modified as well as inequalities for components of Gel’fand-Zetlin scheme in comparison with the same contraction $j_1 = \iota_1$ in basic variant. The eigenvalues of Casimir operators depend not on components $m_{1n}$, $m_{nn}$, as in basic variant, but on other two components of the upper row.

Thus, each of $\binom{n}{2} = n(n - 1)/2$ variants of transition from irreducible representation of algebra $u(n)$ gives under contractions its own irreducible representation of algebra $u(n;j)$, which spectrum of Casimir operators is determined by its own two components of the upper row of Gel’fand-Zetlin scheme. In this case all variants of transition are of general type, i.e. lead to non-zero spectrum of all Casimir operators, even when all parameters $j$ take dual values.

The considerations brought above are valid for each algebra $u(k;j')$, $k = 2, 3, \ldots, n - 1$, in the chain of subalgebras, described in §3.1, i.e. for each subalgebra there are $\binom{k}{2} = k(k - 1)/2$ variants of transition from irreducible representation of subalgebra $u(k)$ to general irreducible
representations of subalgebra \( u(k;j') \). The latter determine Gel’fand-Zetlin basis. Therefore each of \( \binom{n}{2} \) variants of transition from irreducible representation of algebra \( u(n) \) to irreducible representations of algebra \( u(n;j) \) can be written in \( N_{n-1} = \sum_{k=2}^{n-1} \binom{k}{2} \) different bases, corresponding to different variants of transformation of Gel’fand-Zetlin scheme components in the rows with numbers \( k = 2, 3, \ldots, n - 1 \). In the first two sections of this paragraph we have described basic variant, in which the first and the last components of the rows with numbers \( k = 2, 3, \ldots, n \) undergo transformation. It is clear that, if necessary, similar relations can be written for each of \( N_n = \sum_{k=2}^{n} \binom{k}{2} \) variants.

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