Possible quantum kinematics

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The quantum group and space theory is reformulated from the standard skew-symmetric basis to an arbitrary one. The \(N\)-dimensional quantum Cayley–Klein spaces are described in Cartesian basis and the quantum analogs of \((N-1)\)-dimensional constant curvature spaces are introduced. Part of the four-dimensional constant curvature spaces are interpreted as the noncommutative analogs of \((1+3)\) space-time models. As a result the quantum (anti) de Sitter, Minkowski, Newton, Galilei, Carroll kinematics with the fundamental length and the fundamental time are suggested. © 2006 American Institute of Physics. [DOI: 10.1063/1.2157093]

I. INTRODUCTION

Space-time is a fundamental conception that underlines the most significant physical theories. Therefore the analysis of a possible space-time model (or kinematics) has the fundamental meaning for physics. Space and time in nonrelativistic physics were regarded as independent what mathematically is connected with the fiber property of Galilei kinematics. In special relativity it was determined that space and time depend on each other and must be regarded as an integrated object, namely flat Minkowski space-time with a pseudo-Euclidean metric. The notion of curvature was introduced in physics by general relativity. Anti-de Sitter and de Sitter kinematics with constant positive and negative curvature, respectively, are the simplest relativistic space-time models with curvature. Possible kinematics, which satisfy the natural physical postulates: space is isotropic and rotations in space-time planes form a noncompact subgroup, were described in Ref. 1 on the level of Lie algebras. From the point of view of geometry these kinematics are realized as constant curvature spaces, which can be obtained from the spherical space by contractions and analytical continuations known as a Cayley–Klein (CK) scheme.2

The Snyder quantized space-time coordinates or, respectively, the curved momentum space, are the oldest example of using the noncommutative geometry in physics. The simplest curved de Sitter geometry with constant curvature was used instead of flat Minkowski space in different generalizations of quantum field theory as a momentum space model. The universal constant, the fundamental length \(l\), or fundamental mass \(M\), related to \(l\) by \(l = \hbar/Mc\), where \(\hbar\) is the Plank constant and \(c\) is the velocity of light, enters necessarily into the theory.4,6,7

A new possibility for construction of the noncommutative space-time models is provided by quantum groups and quantum vector spaces.8 According to Dirac9 from the early 30s, “The most powerful method of advance that can be suggested at present is to employ all the resources of pure mathematics in attempts to perfect and generalize the mathematical formalism that forms the existing basis of theoretical physics, and after each success in this direction, to try to interpret the new mathematical features in terms of physical entities.” Similar views were expressed by Faddeev,10 where we found this cite. The quantum Poincaré group related to the \(\kappa\)-Poincaré algebra as well as the \(\kappa\)-Minkowski kinematics were suggested.11–13 A general formalism that allows the construction of field theory in \(\kappa\)-Minkowski space-time was developed.14

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On the other hand, the quantum deformations can be useful in the large-scale limit, in particular, for a dark matter problem. If one uses a so-called Maslanka mapping,\textsuperscript{15}

\[
\bar{P}_0 = 2\kappa \text{arcsinh} \frac{P_0}{2\kappa},
\]

where \(P_0 (\bar{P}_0)\) is the energy of the particle for Minkowski (\(\kappa\)-Minkowski) kinematics, then, as it was pointed out by Bacry,\textsuperscript{16} the energy of a system \(S\) composed of two subsystems, \(S(1)\) and \(S(2)\), reads as

\[
2\kappa \sinh \frac{\bar{P}_0}{2\kappa} = 2\kappa \sinh \frac{\bar{P}_{0(1)}}{2\kappa} + 2\kappa \sinh \frac{\bar{P}_{0(2)}}{2\kappa},
\]

rather then \(P_0 = P_{0(1)} + P_{0(2)}\). Actually, the above implies that \(\bar{P}_0 < \bar{P}_{0(1)} + \bar{P}_{0(2)}\). It means that the total energy of the universe is not proportional to the number of particles it contains. Hence there is no need for dark matter due to the kinematical reason.

Our purpose in this paper is to obtain the noncommutative (quantum) analogs of the possible kinematics. It is made just in the same way as for the commutative case, with the exception of the initial Euclidean space, which is substituted by the quantum Euclidean space associated with the quantum orthogonal group. The CK scheme of contractions and analytical continuations was developed in the Cartesian basis, whereas the standard quantum group theory\textsuperscript{8} was formulated in a different skew-symmetric one. Therefore, first of all, this theory is reformulated in the Cartesian basis; then the noncommutative analogs of constant curvature spaces (CCS) including fiber (or flag) spaces are investigated and some of them are interpreted as noncommutative kinematics.

The paper is organized as follows. In Sec. II, we briefly recall the description of the classical commutative kinematics as spaces of constant curvature. In Sec. III, the general formalism of quantum Cayley–Klein orthogonal groups and associated spaces is developed. Section IV is devoted to the investigation of noncommutative four-dimensional space-time models.

II. COMMUTATIVE KINEMATICS

Classical four-dimensional space-time models can be obtained\textsuperscript{2} by the physical interpretation of the orthogonal coordinates of the most symmetric spaces, namely constant curvature spaces. All 3\(N\) \(N\)-dimensional CCS are realized on the spheres,

\[
S_N(j) = \{\xi_1^2 + j_1\xi_2^2 + \ldots + (1,N+1)^2\xi_{N+1}^2 = 1\},
\]

where

\[
(i,k) = \prod_{l=\min(i,k)}^{\max(i,k)-1} j_l, \quad (k,k) = 1,
\]

and each of parameters \(j_k\) takes the values \(1, \xi_k, i, k = 1, \ldots, N\). Here \(\xi_k\) are nilpotent generators \(\xi_k^2 = 0\), with commutative law of multiplication \(\xi_k \xi_m = \xi_m \xi_k = 0, k \neq m\).

The intrinsic Beltrami coordinates \(x_k = \xi_k + \xi_{k+1}\), \(k = 1, 2, \ldots, N\) present the coordinate system in CCS, which coordinate lines \(x_k = \text{const}\) are geodesics. CCS has positive curvature for \(j_1 = 1\), negative for \(j_1 = i\), and it is flat for \(j_1 = -i\). For a flat space the Beltrami coordinates coincide with the Cartesian ones. Nilpotent values \(j_k = i \xi_k, k > 1\) correspond to a fiber (flag) spaces and imaginary values \(j_k = i\) correspond to pseudo-Riemannian spaces.
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Classical (1 + 3) kinematics are obtained from CCS for \( N = 4 \), \( j_1 = 1, t_1, i, j_2 = t_2, i, j_3 = j_4 = 1 \) if one interprets \( x_1 \) as the time axis \( t = \xi_2 \xi_1^{-1} \) and the rest as the space axes \( t_k = \xi_{k+1} \xi_k^{-1}, k = 1, 2, 3 \).

The standard de Sitter kinematics \( \mathbb{S}^N_{(\pm)} \) with constant negative curvature is realized for \( j_1 = j_2 = i, \) anti-de Sitter kinematics \( \mathbb{S}^N_{(\pm)} \) with positive curvature—for \( j_1 = 1, j_2 = i. \) Relativistic flat Minkowski kinematics \( \mathbb{M}_4 \) appears for \( j_1 = t_1, j_2 = i. \) Nonrelativistic Newton \( \mathbb{N}^1_{(\pm)} \) and Galilei \( \mathbb{G}_4 \) kinematics correspond to \( j_2 = t_2, j_1 = t_1, i \) and \( j_2 = t_2, j_1 = t_1, i \) respectively.

If one interprets three first Beltrami coordinates as space axes, while the last one as a time axis, and puts \( j_2 = j_1 = 1, j_3 = t_4, \) then three exotic Carroll kinematics are obtained, namely \( \mathbb{C}_4 \) for \( j_1 = t_4, \) with zero curvature, \( \mathbb{C}_4 \) for \( j_1 = 1, i, \) with positive and negative curvature. The Carroll space and time have contrary properties as compared with those of Galilei kinematics. The Galilei time is absolute, i.e., two events simultaneous in some reference frame remain simultaneous in any reference frame that is obtained by Galilei boost (or space-time rotation) from the initial one. On the contrary, in Carroll kinematics the space is absolute, i.e., two events with the equal spatial coordinates in some reference frame have the same spatial coordinates in any reference frame that is obtained from the initial one by space-time rotation.

III. QUANTUM ORTHOGONAL GROUPS AND QUANTUM CAYLEY–KLEIN SPACES

According to FRT theory, the algebra function on quantum orthogonal group \( \text{Fun}(SO_q(N)) \) or simply quantum orthogonal group \( SO_q(N) \) is the algebra of noncommutative polynomials of \( n^2 \) variables \( t_{ij}, i, j = 1, \ldots, n, \) that are the subject of commutation relations,

\[
R_q T_1 T_2 = T_2 T_1 R_q, \tag{3}
\]

and additional relations of \( q \) orthogonality,

\[
TCT^i = C, \quad T^i C^{-1} T = C^{-1}. \tag{4}
\]

Here \( T_1 = T \otimes I, T_2 = I \otimes T \in M_n (C(t_{ij})) \), \( T = (t_{ij})^{n \times n} \in M_n (C(t_{ij})) \), \( I \) is the unit matrix in \( M_n (C), C = C_{0q} \mathbb{P}, \rho = \text{diag}(\rho_1, \ldots, \rho_N), (C_0)_{ij} = \delta_{ij}, i' = N + 1 - i, \) \( i, j = 1, \ldots, N, \) that is, \( (C)_{ij} = q^{\rho_i \delta_{i'j}} \) and \( C^{-1} = C \),

\[
(\rho_1, \ldots, \rho_N) = \begin{cases} 
\left( n - \frac{1}{2}, n - \frac{3}{2}, \ldots, \frac{1}{2}, 0, -\frac{1}{2}, \ldots, -n + \frac{1}{2} \right), & N = 2n + 1, \\
\left( n - 1, n - 2, \ldots, 1, 0, 0, -1, \ldots, -n + 1 \right), & N = 2n. 
\end{cases} \tag{5}
\]

The numerical matrix \( R_q \) is the well-known solution of Yang–Baxter equation and its elements fulfills the role of the structure constant of quantum group generators.

Let us remind the definition of the quantum vector space. An algebra \( O_q^N(C) \) with generators \( x_1, \ldots, x_N \) and commutation relations

\[
\hat{R}_q(x \otimes x) = q x \otimes x - \frac{q - q^{-1}}{1 + q^{-1}} x' C x W_q, \tag{6}
\]

where \( \hat{R}_q = PR_q, Pu \otimes v = v \otimes u, \forall u, v \in C^n, W_q = \sum_{i=1}^n q^{\rho_i} e_i \otimes e_i^\dagger, \)

\[
x' C x = \sum_{i,j=1}^N x_i C_{ij} x_j = \epsilon x_{n+1}^2 + \sum_{k=1}^n (q^{\rho_k} x_k x_{k+1} + q^{\rho_k} x_k x_{k-1}), \tag{7}
\]

\( \epsilon = 1 \) for \( N = 2n + 1, \) \( \epsilon = 0 \) for \( N = 2n \) and vector \( (e_i)_k = \delta_{ik}, \) \( i, k = 1, \ldots, N \) is called the algebra of functions on \( N \)-dimensional quantum Euclidean space (or simply the quantum Euclidean space) \( O_q^N(C) \).

The coaction of the quantum group \( SO_q(N) \) on the noncommutative vector space \( O_q^N(C) \) is given by
\[ \delta(x) = T \otimes x, \quad \delta(x_i) = \sum_{k=1}^{n} t_{ik} \otimes x_k, \quad i = 1, \ldots, n, \]  

and quadratic form (7) is invariant inv = x^T C x with respect to this coaction.

The matrix C has nonzero elements only on the secondary diagonal. They are equal to unity in the commutative limit \( q=1 \). Therefore the quantum group SO\(_q(N)\) and the quantum vector space \( O_q^N(\mathbb{C}) \) are described by equations (3), (4), (6), (7) in a skew-symmetric basis, where for \( q=1 \) the invariant form inv = x^T C x is given by the matrix \( C_0 \) with the only nonzero elements on the secondary diagonal that are all equal to real units.

New generators \( y = D^{-1} x \) of the vector space \( O_q^N(\mathbb{C}) \) in arbitrary basis are obtained \(^{18,19}\) with the help of nondegenerate matrix \( D \in M_N \) and they are subject of the commutation relations,

\[ \hat{R}(y \otimes y) = q y \otimes y - \frac{\lambda}{1 + q} y^T C y W, \]  

where \( \hat{R} = (D \otimes D)^{-1} \hat{R}_q(D \otimes D), \quad W = (D \otimes D)^{-1} W_q, \quad C' = D^T C D. \) The corresponding quantum group \( SO_q(N) \) is generated in arbitrary basis by \( U = (u_{ij})_{i,j=1}^{N} \), where \( U = D^{-1} T D \). The commutation relations of the new generators are

\[ \hat{R} U_i U_j = U_j U_i \hat{R} \]  

and \( q \)-orthogonality relations look as follows:

\[ U C U = \hat{C}, \quad U(\hat{C})^{-1} U = (\hat{C})^{-1}, \]  

where \( \hat{R} = (D \otimes D)^{-1} R_q(D \otimes D), \quad \hat{C} = D^{-1} C(D^{-1})' \).

In the case of kinematics, the most natural basis is the Cartesian basis, where the invariant form inv = y^T y is given by the unit matrix I. The transformation from the skew-symmetric basis x to the Cartesian basis y is described by the matrix D, which is a solution of the following equation:

\[ D^T C_0 D = I. \]  

This equation has many solutions. Take one of these, namely

\[ D = \frac{1}{\sqrt{2}} \begin{pmatrix} I & 0 & -i \hat{C}_0 \\ 0 & \sqrt{2} & 0 \\ \hat{C}_0 & 0 & iI \end{pmatrix}, \quad N = 2n + 1, \]  

where \( \hat{C}_0 \) is the \( n \times n \) matrix with real units on the secondary diagonal. For \( N = 2n \) the matrix D is given by (13) without the middle column and row. The matrix (13) provides one of the possible combinations of the quantum group structure and the CK scheme of group contractions. All other similar combinations are given by the matrices \( D_\sigma = D V_\sigma \) obtained from (13) by the right multiplication on the matrix \( V_\sigma \in M_N \) with elements \( (V_\sigma)_{ik} = \delta_{\sigma k}, \) where \( \sigma = S(n) \) is a permutation of the Nth order. \(^{20}\) The matrices \( D_\sigma \) are solutions of Eq. (12).

We derive the quantum Cayley–Klein spaces with the same transformation of the Cartesian generators \( y = \psi \xi, \quad \psi = \text{diag}[1, (1,2), \ldots, (1, N)] \in M_N, \) as in the commutative case. \(^2,19\) The transformation \( z = \text{Ju} \) of the deformation parameter \( q = e^z \) should be added in the quantum case. The commutation relations of the Cartesian generators of the quantum N-dimensional Cayley–Klein space are given by the equations

\[ \hat{R}_\sigma(j) \xi \otimes \xi = e^{j_\xi} \xi \otimes \xi - \frac{2 \chi \text{Ju}}{1 + e^{j_\text{N-i}\xi}} \xi^T C_\sigma(j) \xi W_\sigma(j), \]  

where
\[ \hat{R}_\sigma(j) = \Psi^{-1} \hat{R}_\sigma \Psi, \quad W_\sigma(j) = \Psi^{-1} W_\sigma, \]

and in explicit form are
\[ [\xi_{\sigma_k}, \xi_{\sigma_m}] = \xi_{\sigma_m} \xi_{\sigma_k} \cosh jv - i \xi_{\sigma_m} \xi_{\sigma_k} (1, \sigma_k) (1, \sigma_m)^{-1} \sinh jv, \quad k < m, \quad k \neq m', \]
\[ \xi_{\sigma_k} \xi_{\sigma_m} = \xi_{\sigma_m} \xi_{\sigma_k} \cosh jv - i \xi_{\sigma_m} \xi_{\sigma_k} (1, \sigma_m) (1, \sigma_k)^{-1} \sinh jv, \quad m' < m, \quad k \neq m', \]
\[ [\xi_{\sigma_k}, \xi_{\sigma_m}] = 2i \epsilon \sinh \left( \frac{jv}{2} \right) \left( \cosh jv \right)^n \xi_{\sigma_n} \left( 1, \sigma_k \right)^n \left( 1, \sigma_n \right)^{-1} \sinh jv, \quad k < m, \quad k \neq m', \]
\[ \left( [\xi_{\sigma_k}, \xi_{\sigma_m}] \right) = \frac{\sinh(jv)}{(\cosh(jv))^{k+1}(1, \sigma_k)(1, \sigma_m)} \sum_{m=1}^n (\cosh(jv))^m \left[ (1, \sigma_m)^2 \xi_{\sigma_m}^2 + (1, \sigma_m)^2 \xi_{\sigma_m}^2 \right], \]

where \( k, m = 1, 2, \ldots, n \). The invariant form of the Cayley–Klein space \( O^N(j; \sigma; C) \) is written as
\[ \text{inv}(j) = \cosh(jv) \rho_1 \left( \epsilon(1, \sigma_{n+1})^2 \xi_{\sigma_n} \cosh(jv/2) + \sum_{k=1}^n ((1, \sigma_k)^2 \xi_{\sigma_k}^2 + (1, \sigma_k)^2 \xi_{\sigma_k}^2) \cosh(jv)^{k-1} \right). \]

The multiplier \( J \) in the transformation \( z = jv \) of the deformation parameter is chosen as \( J = \bigcup_{k=1}^n (\sigma_k, \sigma_k) \). This is the minimal multiplier, which guarantees the existence of the Hopf algebra structure for the associated quantum group \( SO_\sigma(N; j; \sigma) \). The “union” \( (\sigma_k, \sigma_k) \cup (\sigma_m, \sigma_m) \) is understood as the first power multiplication of all parameters \( j_{\sigma_k} \), which occur at least in one multiplier \( (\sigma_k, \sigma_k) \) or \( (\sigma_m, \sigma_m) \), for example, \( (j_{\sigma_k} j_{\sigma_m}) (j_{\sigma_m} j_{\sigma_k}) = j_{\sigma_k} j_{\sigma_m} \).

The quantum orthogonal Cayley–Klein sphere \( S^N(j; \sigma) \) is obtained as the quotient of \( O^N(j; \sigma) \) by \( \text{inv}(j) = 1 \). The quantum analogs of the intrinsic Beltrami coordinates on this sphere are given by the sets of independent right or left generators,
\[ r_{\sigma_k}^{-1} = \xi_{\sigma_k} \xi_{\sigma_k}^{-1}, \quad r_{\sigma_k}^{-1} = \xi_{\sigma_k} \xi_{\sigma_k}^{-1}, \quad i = 1, \ldots, N, \quad i \neq k, \quad \sigma_k = 1. \]

In the case of quantum Euclidean spaces \( O^N_q(C) \), the use of different \( D_\sigma \) for \( \sigma \in S(N) \) makes no sense, because all similarly obtained quantum spaces are isomorphic. However, the situation is radically different for the quantum Cayley–Klein spaces. In this case the Cartesian generators \( \xi_k \) are multiplied by \( (1, k) \) and for nilpotent values of all or some parameters \( j_k \) this isomorphism of quantum vector spaces is destroyed. The necessity of using different \( D_\sigma \) arises as well if there is some physical interpretation of generators. In this case physically different generators may be confused by permutations \( \sigma \), for example, time and space generators of kinematics. Mathematically isomorphic kinematics may be physically nonequivalent.

**IV. QUANTUM KINEMATICS**

For \( N=5 \) the thorough analysis of the multiplier \( J = (\sigma_1, \sigma_5) \cup (\sigma_2, \sigma_4) \), which appears in the transformation of the deformation parameter \( z = jv \), and commutation relations (15) of the quantum space generators for different permutations allowed to find three permutations giving different \( J \) and a physically nonequivalent kinematics, namely \( \sigma_5 = (1, 2, 3, 4, 5) \), \( \sigma^* = (1, 4, 3, 5, 2), \) \( \sigma^* = (2, 3, 1, 4, 5). \)

In order to clarify the relation with the standard Inonu–Wigner contraction procedure, the mathematical parameter \( j_1 \) is replaced by the physical one \( j_1 T^{-1} \), and the parameter \( j_2 \) is replaced
by $ic^{-1}$, where $j_1 = 1, i$. The limit $T \to \infty$ corresponds to the contraction $j_1 = t_1$, and the limit $c \to \infty$ corresponds to $j_2 = c_2$. The parameter $T$ is interpreted as the curvature radius and has the physical dimension of time $[T] = [\text{time}]$, the parameter $c$ is the light velocity $[c] = [\text{length}] \times [\text{time}]^{-1}$.

As far as the generator $\xi_i$ does not commute with others, it is convenient to introduce right and left time $t = \xi_j \xi_i^{-1}$, $\tilde{t} = \xi_i \xi_j^{-1}$ and space $r_j = \xi_k \xi_j^{-1}$, $\tilde{r}_j = \xi_j \xi_k^{-1}$, $k = 1, 2, 3$, generators. The reason for this definition is the simplification of expressions for commutation relations of quantum kinematics. It is possible to use only, say, right generators, but its commutators are cumbersome in the case of the (anti-)de Sitter kinematics. The commutation relations of the independent generators are obtained (see Ref. 22 for details) in the form

$$S^{(s)}_u(r_0) = \left\{ t, r \ \hat{r}_1 = \hat{r}_1 t \cos \frac{\vec{j}_1 u}{cT} + i \hat{r}_1 r_2 \frac{1}{c} \sin \frac{\vec{j}_1 u}{cT}, \hat{r}_2 t = -2i \hat{r}_1 r_1 \frac{1}{c} \sin \frac{\vec{j}_1 u}{2cT}, \hat{r}_3 t = \hat{r}_3 t \cos \frac{\vec{j}_1 u}{cT}, \right.$$

$$- i \hat{r}_1 \frac{cT}{j_1} \sin \frac{\vec{j}_1 u}{cT}, \hat{r}_2 r_2 = \hat{r}_2 r_1 \cos \frac{\vec{j}_1 u}{cT} - i \hat{r}_1 c \sin \frac{\vec{j}_1 u}{cT}, \hat{r}_p r_3 = \hat{r}_p r_1 \cos \frac{\vec{j}_1 u}{cT} - i r_p \frac{cT}{j_1} \sin \frac{\vec{j}_1 u}{cT},$$

(18)

where the right and left generators are connected as follows:

$$r_3 - \hat{r}_3 = 2 \frac{\vec{j}_1 u}{cT} \left( \left( \frac{i}{c^2} \hat{r}_2 r_2 \right) \cos \frac{\vec{j}_1 u}{2cT} - i \frac{1}{c^2} \hat{r}_1 r_1 \cos \frac{\vec{j}_1 u}{cT} \right) \sin \frac{\vec{j}_1 u}{2cT},$$

$$\hat{r}_p = r_p \cos \frac{\vec{j}_1 u}{cT} - i \hat{r}_p r_1 \sin \frac{\vec{j}_1 u}{cT}, \ p = 1, 2,$$

(19)

$$\hat{t} = t \cos \frac{\vec{j}_1 u}{cT} - i \hat{r}_1 \frac{\vec{j}_1 u}{cT} \sin \frac{\vec{j}_1 u}{cT}. $$

$$S^{(s)}_u(r') = \left\{ t, r \right\} \hat{r}_k = i r_k \left( \frac{\vec{j}_1 u}{T} - i r_k \right) \sinh \frac{\vec{j}_1 u}{j_1}, \hat{r}_2 r_1 = \hat{r}_2 r_1 \cosh \frac{\vec{j}_1 u}{T} - i \hat{r}_2 r_3 \sinh \frac{\vec{j}_1 u}{T}, \hat{r}_3 r_2 = \hat{r}_3 r_2 \cosh \frac{\vec{j}_1 u}{T} - i \hat{r}_3 r_1 \sinh \frac{\vec{j}_1 u}{T} = \hat{r}_3 r_1 \cosh \frac{\vec{j}_1 u}{T} - i \hat{r}_2 r_1 \sinh \frac{\vec{j}_1 u}{T}, \hat{r}_2 r_3 - \hat{r}_3 r_2 = 2 \hat{r}_1 r_1 \sinh \frac{\vec{j}_1 u}{2T},$$

(20)

where the right and left generators are connected as

$$\hat{r}_k = r_k \cosh \frac{\vec{j}_1 u}{T} + i r_k \frac{\vec{j}_1 u}{T} \sinh \frac{\vec{j}_1 u}{T},$$

(21)

$$\hat{t} = t + 2i \frac{\vec{j}_1 u}{c^2 T} \left( \hat{r}_1 r_1 \cosh \frac{\vec{j}_1 u}{T} + \left( \hat{r}_2 r_2 + \hat{r}_3 r_3 \right) \cosh \frac{\vec{j}_1 u}{2T} \right) \sinh \frac{\vec{j}_1 u}{2T}.$$
the fundamental length. For the permutation \( /H_9268 \), \( /H_20849 \)

\[ r_k = r_3, \quad r_3 = r_1 \cos \frac{v}{c} + i c \sin \frac{v}{c}. \]  

(23)

In the case of the identical permutation \( \sigma_0 \), deformation parameter \( v \) for the system units, where \( \hbar = 1 \), has the physical dimension of length \([v]=cT=\text{[length]}\) and may be interpreted as the fundamental length. For the permutation \( \sigma' \), the quantum (anti-) de Sitter kinematics (20) are characterized by the fundamental time \([v]=\text{[time]}\) and for the permutation \( \tilde{\alpha} \) are characterized by the fundamental velocity \([v]=\text{[velocity]}\). Recall that the same physical dimensions of the deformation parameter have been obtained for the quantum algebras \( so_v(3; j; \sigma) \) and corresponding \((1+1)\) kinematics for different permutations.23

As it follows from (22), (23), both contractions \( T \to \infty, c \to \infty \) are not permitted, therefore the quantum (anti-) de Sitter kinematics \( S_v^{so_v(3; j; \sigma)} \) do not have Minkowski, Newton, and Galilei kinematics as limiting cases.

In the zero curvature limit \( T \to \infty \) two quantum Minkowski kinematics are obtained,

\[ M_v^4(\sigma_0) = \{ t, r \}, \quad [t, r] = 0, \quad [r_3, r] = iv, \quad [r_2, r_1] = 0, \quad [r_3, r_p] = ivr_p, \quad p = 1, 2, \}, \]

\[ M_v^4(\sigma') = \{ t, r \}, \quad [t, r_3] = ivr_3, \quad [r_2, r_1] = 0, \quad i, k = 1, 2, 3. \]  

(24)

The first one is isomorphic to the tachyonic \( \kappa \)-Minkowski kinematics, the second one to the standard \( \kappa \)-deformation.11–13 For both \( \kappa \)-Minkowski kinematics in the system units \( \hbar = c = 1 \), the deformation parameter \( \Lambda = \kappa^{-1} \) has the physical dimension of length and is interpreted as the fundamental length. But in the system units \( \hbar = 1 \), the deformation parameter has different dimensions, namely \( v \) is the fundamental length for \( M_v^4(\sigma_0) \) and \( v \) is the fundamental time for \( M_v^4(\sigma') \).

As far as the commutation relations (24) do not depend on \( c \), they do not change in the limit \( c \to \infty \), therefore the generators of the quantum Galilei kinematics \( G_v^4(\sigma_0) \) and \( G_v^4(\sigma') \) are the subject of the same commutation relations. The only difference consists in the following statement: for the Galilei kinematics there are two invariants \( \text{inv}_1 = t^2, \text{inv}_2 = r_1^2 + r_2^2 + r_3^2 \) with respect to the coaction of the corresponding quantum groups, whereas for the Minkowski kinematics there is only one invariant \( \text{inv} = t^2 - (r_1^2 + r_2^2 + r_3^2) \). Thereby the quantum deformations of the flat kinematics are identical up to the coaction of the corresponding quantum groups for both relativistic and nonrelativistic kinematics.

In the nonrelativistic limit \( c \to \infty \) there are two noncommutative analogs of the Newton kinematics:

\[ N_v^{4(\sigma_0)} = \{ t, r \}, \quad [t, r_p] = 0, \quad [r_3, t] = ivt, \quad [r_2, r_1] = 0, \quad [r_3, r_p] = ivr_p, \quad p = 1, 2, \}

\[ N_v^{4(\sigma')} = \{ t, r \}, \quad [t, r_3] = ivr_3, \quad [r_2, r_1] = 0, \quad i, k = 1, 2, 3. \]  

(25)

where in the last case the deformation parameter is not transformed under contraction. The mul-
tiplier $T^{-1}$ appears as the result of the physical interpretation of the quantum space generators. For nonzero curvature kinematics commutation relations of generators depend on $c$ and are different for relativistic and nonrelativistic cases, unlike Minkowski and Galilei kinematics.

The exotic Carroll kinematics are also realized as constant curvature spaces, but with different interpretation of the Beltrami coordinates, namely $r_k = \xi_{k+1} \xi_1^{-1}$, $k=1,2,3$ are the space generators and $t = \xi_0 \xi_1^{-1}$ is the time generator. Due to this interpretation the new physical dimensions of the contraction parameters appear: the parameter $j_1$ is replaced by $j_kR^{-1}$, where $R \to \infty$ corresponds to $j_1 = t_4$, and $[R]= [\text{length}]$; the parameter $j_4$ is replaced by $c$, where $c \to 0$ corresponds to $j_4 = t_4$ and $[c] = [\text{velocity}]$. There are three noncommutative analogs of the exotic nonzero curvature Carroll kinematics:

$$C_v^{(a)}(\sigma_0) = \left\{ t, r_i[t_i, r_k] = ivr_i \left(1 + \frac{j_1}{R^2} r^2 \right), [r_i, r_k] = 0 \right\},$$

$$C_v^{(a)}(\sigma') = \left\{ t, r_i[t_i, r_k] = 0, [r_i, r_k] = 0, [r_3, t] = i v j_1 \frac{r_1}{R^2}, [r_1, r_2] = 0 \right\},$$

$$C_v^{(p)}(\sigma) = \left\{ t, r_i[t_i, r_k] = 0, [r_i, r_k] = 0, [r_1, r_2] = i v \left( \frac{R^2}{j_1} r + \frac{r^2}{j_1} \right) \right\}.$$  

Two quantum analogs of the zero curvature Carroll kinematics are achieved in the limit $R \to \infty$ and are as follows:

$$C_v^{(a)}(\sigma_0) = \left\{ t, r_i[t_i, r_k] = ivr_i, [r_i, r_k] = 0, i, k = 1,2,3 \right\},$$

$$C_v^{(a)}(\sigma') = \left\{ t, r_i[t_i, r_k] = 0, [r_i, r_k] = 0, i, k = 1,2,3 \right\}.  \tag{27}$$

For the permutations $\sigma_0$, $\sigma'$ the deformation parameter $v = Rc^{-1}$ has the physical dimension of time $[v] = [\text{time}]$ and is interpreted as the fundamental time. For the permutation $\sigma$ the deformation parameter $v = c^{-1}$ has the dimension of inverse velocity $[v] = [\text{velocity}]^{-1}$ and may be interpreted as the fundamental velocity.

V. CONCLUSION

We have reformulated the quantum orthogonal group $SO_q(N)$ and the corresponding $q$-Euclidean space $O_q^N$ in Cartesian coordinates and then used the standard trick with real, complex, and dual numbers in order to define the quantum Cayley–Klein spaces of constant curvature $O_q^N (j; \sigma)$ uniformly, using a $q$ analog of Beltrami coordinates. The different combinations of quantum structure and CK scheme of contractions and analytical continuations are described with the help of permutations $\sigma$. As a result, for $N=5$, the quantum deformations of (anti-) de Sitter, Minkowski, Newton, Galilei, and Carroll kinematics are obtained.

We have found two types of the noncommutative realistic space-time models with fundamental length and fundamental time, which admit nonrelativistic and zero curvature limits and one type of the (anti-) de Sitter kinematics with fundamental velocity, where both limits are forbidden. For the exotic Carroll kinematics there are two types with fundamental time, which admit zero curvature limit and one type with fundamental velocity, where this limit is forbidden.

The quantum Galilei kinematics $G_q^A(\sigma_0)$ and $G_q^A(\sigma')$ have the same commutation relations (24) as the quantum Minkowski kinematics $M_q^A(\sigma_0)$ and $M_q^A(\sigma')$. In other words, the quantum deformations of the flat kinematics are identical up to the coaction of the corresponding quantum groups, whereas for nonzero curvature kinematics commutation relations of generators are different for relativistic and nonrelativistic cases.
In spite of the fact that the commutation relations of generators of Carroll $C^d(0)$ ($\sigma_i$) (27) and Minkowski $M^d(\sigma')$ (24) kinematics are identical, both kinematics are physically different. Mathematically isomorphic kinematics may be physically nonequivalent.

Noncommutative kinematics are obtained by the interpretation of some mathematical constructions associated with quantum groups and quantum spaces. The deformation parameter is free parameter of these models. Which type of model is more appropriate and what is the value of deformation parameter are questions of experimental study.

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