ON CONTRACTIONS OF QUANTUM ORTHOGONAL GROUPS

N.A. Gromov, I.V. Kostyakov, V.V. Kuratov
Department of Mathematics, Syktyvkar Branch of IMM,
Kommunisticheskaya st. 24, Syktyvkar, 167000, Russia
E-mail: gromov@dm.komisc.ru

Abstract

Instead of zero tending parameters of Wigner–Inönü our approach to a group contractions is based on use of the nilpotent commutative generators of Pimenov algebra \( \mathbf{D}(\iota) \), which is a subalgebra of even part of Grassmann algebra. The standard Faddeev quantization of the simple groups is modified in such a way that the quantum analogs of the nonsemisimple groups are obtained by contractions.

The contracted quantum groups are regarded as the algebras of noncommutative functions generated by elements \( J_{ik} t_{ik} \), where \( J_{ik} \) are some products of generators of the algebra \( \mathbf{D}(\iota) \) and \( t_{ik} \) are the noncommutative generators of quantum group. Possible contractions of quantum orthogonal groups are regarded in detail. They essentially depend on the choice of primitive elements of the Hopf algebra. All such choices are considered for quantum group \( SO_q(N; C) \) and all allowed contractions in Cayley–Klein scheme are described.

The quantum deformations of the complex kinematical groups have been investigated as a contractions of \( SO_q(5; C) \) and have shown that the result is connected with the behavior of deformation parameter under contraction. If deformation parameter \( q \) remain unchanged, then the quantum Euclidean \( E_q(4; C) \) and Newton \( N_q(4; C) \) groups are obtained. If the deformation parameter is transformed, then one more nonisomorphic quantum deformation of Newton group \( N_q(4; C) \) is obtained. But there is no quantum analog of the (complex) Galilei group \( G(1, 3) \) in both cases.

According to correspondence principle a new physical theory must include an old one as a particular case. For space-time symmetries this principle is realized as the chain of contractions of the kinematical groups:

\[
S^\pm(1, 3) \xrightarrow{K \to 0} P(1, 3) \xrightarrow{c \to \infty} G(1, 3).
\]

As it was mentioned above there is no quantum deformation of the complex Galilei group in the standard Cayley–Klein scheme, therefore it is not possible to construct the quantum analog of the full chain of contractions of the (1+3) kinematical groups even at the level of a complex groups.


1 Introduction

Contraction of Lie groups (algebras) is the method of obtaining a new Lie groups (algebras) from some initial one’s with the help of passage to the limit [1]. One may define contraction of algebraic structure \((M, \ast)\) as a map \(\phi_\epsilon : (M, \ast) \to (N, \ast')\), where \((N, \ast')\) is the algebraic structure of the same type, isomorphic to \((M, \ast)\) for \(\epsilon \neq 0\) and nonisomorphic to the initial algebraic structure for \(\epsilon = 0\). Except for Lie group (algebra) contractions, graded contractions \([2]\), \([3]\) are known, which preserve the grading of Lie algebra. Under contractions of bialgebra \([4]\) Lie algebra structure and cocommutator are conserved. Hopf algebra (or quantum group \([5]\)) contractions are introduced (on the level of quantum algebra \([6]\), \([7]\) and on the level of quantum group \([11]\)) in such a way that in the limit \(\epsilon \to 0\) a new expressions for coproduct, counit and antipode are consistent with the Hopf algebra axioms. Recently contractions of the algebraic structures with bilinear products of arbitrary nature on sections of finite-dimensional vector bundles was presented \([8]\) and contractions of Lie algebroids and Poisson brackets was given as an example.

Low dimensional quantum groups have been studied in details. The two-dimensional Euclidean quantum group \(E_q(2)\) was obtained by contractions of the unitary quantum group \(SU_q(2)\) with untouched deformation parameter \(q\) in \([9]\), \([10]\), \([11]\) and by contractions of the orthogonal quantum group \(SO_q(3)\) with transformed deformation parameter in \([12]\)–\([14]\). The quantum Heisenberg group \(H_q(1)\) was regarded as a contraction of \(SU_q(2)\) in \([15]\) and of \(SO_q(3)\) in \([12]\). A contraction procedure starting from \(SO_q(4)\) was used in \([7]\) to determine \(E_q(3)\). A contraction of the de Sitter quantum group leading to a Poincare quantum group in any dimensions was proposed in \([16]\). Quantum deformations of the inhomogeneous Lie groups have been studied in any dimensions by using the projective (not contraction) method of \([17]\), \([18]\) for the multiparametric quantum groups as well \([19]\), \([20]\). On the other hand \(SO_q(3)\) and \(SO_q(4)\) are not typical representatives of the quantum orthogonal groups \(SO_q(N)\) for \(N = 2n + 1\) and \(N = 2n\), respectively. We shall see that the number of the allowed contractions for \(SO_q(N)\) with the transformed deformation parameter is less then the whole number of the contraction parameters. \(SO_q(3)\) and \(SO_q(4)\) quantum groups are an exceptions, because both such numbers are equal (two and three, respectively). Therefore, the investigation of the contractions of the quantum orthogonal groups \(SO_q(N)\) for an arbitrary \(N\) seems to worth attention. In present paper contractions of the standard deformed quantum group \(SO_q(N)\) \([5]\) are studied in the Cayley–Klein scheme. The preliminary results for the
particular case of identical permutation was published in [21].

Contractions as a passage to limit are corresponded with a physical intuition. At the same time it is desirable to investigate contractions of an algebraic structures with the help of pure algebraic tools. Sometimes it facilitate an investigations, especially in complicate cases. It is possible for classical and quantum Lie groups and algebras if one take into consideration an algebra $\mathbf{D}(\iota)$ with nilpotent commutative generators. In particular, a motion groups of a constant curvature spaces (or Cayley-Klein groups) may be obtained from a classical orthogonal group by replacement its matrix elements with the specific elements of the algebra $\mathbf{D}(\iota)$ [22]. It is worth to note, that at any stage one may to come back to the standart Inönü–Wigner contraction by putting an appropriate parameter tends to zero instead of takes nilpotent value. In present paper the groups under consideration are regarded according to [5] as an algebra of noncommutative functions, but with nilpotent generators. Possible contractions are essentially depended on the choice of primitive elements of Hopf algebra. We have regarded all variants of such choice for the quantum orthogonal group $\text{SO}_q(N)$ and for each variant have found all admissible contractions in Cayley-Klein scheme.

The paper is organized as follows. In Sec. 2, we briefly recall the matrix realizations of the non-quantum orthogonal Cayley-Klein groups both in Cartesian and symplectic bases. In Sec. 3, the formall definition of the quantum complex group $\text{SO}_v(N; j; \sigma)$ is given and analysed when the presented structure of the Hopf algebra is well defined and consistent under nilpotent values of parameters $j_k$. The results are collected in Theorem 1-4. The developed approach is applied to the quantum complex kinematic groups in Sec. 4. The explicit expressions of antipode, coproduct and relations of $(q, j)$-orthogonality for $\text{SO}_v(N; \sigma; j)$ are presented in Appendices A–C. We do not pretend to the fullness of the bibliography. Accessible to us references are included.

2 Orthogonal Cayley-Klein groups

Let us define 

Pimenov algebra $\mathbf{D}_n(\iota; \mathbb{C})$ as an associative algebra with unit over complex number field and with nilpotent commutative generators $\iota_k$, $\iota_k^2 = 0$, $\iota_k \iota_m = \iota_m \iota_k \neq 0$, $k \neq m$, $k, m = 1, \ldots, n$. The general element of $\mathbf{D}_n(\iota; \mathbb{C})$ is in the form

$$d = d_0 + \sum_{p=1}^{n} \sum_{k_1 < \ldots < k_p} d_{k_1 \ldots k_p} \iota_{k_1} \ldots \iota_{k_p}, \quad d_0, d_{k_1 \ldots k_p} \in \mathbb{C}.$$
For $n = 1$ we have $D_1(t_1; \mathbb{C}) \ni d = d_0 + d_1 t_1$, i.e. the elements $d$ are dual (or Study) numbers when $d_0, d_1 \in \mathbb{R}$. For $n = 2$ the general element of $D_2(t_1, t_2; \mathbb{C})$ is $d = d_0 + d_1 t_1 + d_2 t_2 + d_12 t_1 t_2$. Two elements $d, \tilde{d} \in D_n(t; \mathbb{C})$ are equal if and only if $d_0 = \tilde{d}_0, d_{k_1...k_p} = \tilde{d}_{k_1...k_p}, p = 1,\ldots,n$. If $d = d_k t_k$ and $\tilde{d} = \tilde{d}_k t_k$, then the condition $d = \tilde{d}$, which is equivalent to $d_k t_k = \tilde{d}_k t_k$, make possible the consistently definition of the division of nilpotent generator $t_k$ by itself, namely: $t_k/t_k = 1, k = 1,\ldots,n$. Let us stress that the division of different nilpotent generators $t_k/t_p$, $k \neq p$, as well as the division of complex number by nilpotent generators $a/t_k$, $a \in \mathbb{C}$ are not defined. It is convenient to regard the algebras $D_n(j; \mathbb{C})$, where the parameters $j_k = 1, t_k, k = 1,\ldots,n$. If $m$ parameters are nilpotent $j_s = t_s, s = 1,\ldots,m$ and the other are equal to unit, then we have Pimenov algebra $D_m(\epsilon; \mathbb{C})$.

Complex orthogonal Cayley-Klein group $SO(N; j; \mathbb{C})$ is defined as the group of transformations $\xi^t(j) = A(j)\xi(j)$ of complex vector space $O_N(j)$ with Cartesian coordinates $\xi^t(j) = (\xi_1, (1, 2)\xi_2,\ldots,(1, N)\xi_N)^t$, which preserve the quadratic form

$$inv(j) = \xi^t(j)\xi(j) = \xi_1^2 + \sum_{k=2}^{N} (1, k)^2 \xi_k^2,$$

where $j = (j_1,\ldots,j_{N-1})$, each parameter $j_k$ takes two values: $j_r = 1, t_r, r = 1,\ldots,N - 1, \xi_k \in \mathbb{C}$ and

$$(\mu, \nu) = \prod_{l = \min(\mu, \nu)}^{\max(\mu, \nu) - 1} j_l, \quad (\mu, \mu) = 1.$$

Let us stress, that Cartesian coordinates of $O_N(j)$ are special elements of Pimenov algebra $D_{N-1}(j; \mathbb{C})$. It worth notice that the orthogonal Cayley-Klein groups as well as the unitary and symplectic Cayley-Klein groups have been regarded in [23] as the matrix groups with the real matrix elements. Nevertheless there is a different approach, which gives the same results for ordinary groups, but is more appropriate from the contraction quantum group point of view. According with this approach, the Cayley-Klein group $SO(N; j; \mathbb{C})$ may be realised as the matrix group, whose elements are taken from algebra $D_{N-1}(j; \mathbb{C})$ and in Cartesian basis consist of the $N \times N$ matrices $A(j)$ with elements

$$(A(j))_{kp} = (k, p)a_{kp}, \quad a_{kp} \in \mathbb{C}.$$

Matrices $A(j)$ are subject of the additional $j$-orthogonality relations

$$A(j)A^t(j) = A^t(j)A(j) = I.\quad (1)$$
Sometimes it is convenient to regard an orthogonal group in so-called "symplectic" basis. Transformation from Cartesian to symplectic basis $x(j) = D\xi(j)$ is made by unitary matrices $D$, which are solutions of equation

\[ D^tC_0D = I, \quad (2) \]

where $C_0 \in M_N$, $(C_0)_{ik} = \delta_{ik'}$, $k' = N + 1 - k$. To obtain all solutions of equation (2), take one of them, namely

\[
D = \frac{1}{\sqrt{2}} \begin{pmatrix} I & -i\hat{C}_0 \\ \hat{C}_0 & iI \end{pmatrix}, \quad N = 2n, \\
D = \frac{1}{\sqrt{2}} \begin{pmatrix} I & 0 & -i\hat{C}_0 \\ 0 & \sqrt{2} & 0 \\ \hat{C}_0 & 0 & iI \end{pmatrix}, \quad N = 2n + 1, \quad (3)
\]

where $n \times n$ matrix $\hat{C}_0$ is like $C_0$, then regard the matrix $D_{\sigma} = DV_{\sigma}$, $V_{\sigma} \in M_N$, $(V_{\sigma})_{ik} = \delta_{\sigma,k}$, and $\sigma \in S(N)$ is a permutation of the $N$th order. It is easy to verify that $D_{\sigma}$ is again a solution of equation (2). Then in symplectic basis the orthogonal Cayley Klein group $SO(N; j; \mathbb{C})$ is described by the matrices

\[ B_{\sigma}(j) = D_{\sigma}A(j)D_{\sigma}^{-1} \quad (4) \]

with the additional relations of $j$-orthogonality

\[ B_{\sigma}(j)C_0B_{\sigma}^t(j) = B_{\sigma}^t(j)C_0B_{\sigma}(j) = C_0. \]

It should be noted that for orthogonal groups ($j = 1$) the use of different matrices $D_{\sigma}$ makes no sense because all Cartesian coordinates of $O_N$ are equivalent up to a choice of its enumerations. The different situation is for Cayley-Klein groups ($j \neq 1$). Cartesian coordinates $(1, k)\xi_k$, $k = 1, \ldots, N$ for nilpotent values of some or all parameters $j_k$ are different elements of the algebra $D_{N-1}(j; \mathbb{C})$, therefore the same group $SO(N; j; \mathbb{C})$ may be realized by matrices $B_{\sigma}$ with a different disposition of nilpotent generators among their elements. Namely this fact will provide us with different sets of primitive elements of Hopf algebra in the case of quantum group.
Matrix elements of $B_{\sigma}(j)$ are as follows

\[(B_{\sigma})_{n+1,n+1} = b_{n+1,n+1},\]
\[(B_{\sigma})_{kk} = b_{kk} + i\tilde{b}_{kk}(\sigma_k,\sigma_{k'}), \quad (B_{\sigma})_{k'k'} = b_{kk} - i\tilde{b}_{kk}(\sigma_k,\sigma_{k'}),\]
\[(B_{\sigma})_{k'k} = b_{k'k} - i\tilde{b}_{k'k}(\sigma_k,\sigma_{k'}), \quad (B_{\sigma})_{k'k} = b_{k'k} + i\tilde{b}_{k'k}(\sigma_k,\sigma_{k'}),\]
\[(B_{\sigma})_{k,n+1} = b_{k,n+1}(\sigma_k,\sigma_{n+1}) - i\tilde{b}_{k,n+1}(\sigma_{n+1},\sigma_k),\]
\[(B_{\sigma})_{k',n+1} = b_{k',n+1}(\sigma_{k'},\sigma_{n+1}) + \tilde{b}_{k',n+1}(\sigma_{n+1},\sigma_{k'}),\]
\[(B_{\sigma})_{n+1,k} = b_{n+1,k}(\sigma_{n+1},\sigma_k) + \tilde{b}_{n+1,k}(\sigma_{n+1},\sigma_k),\]
\[(B_{\sigma})_{n+1,k'} = b_{n+1,k}(\sigma_{n+1},\sigma_{k'}) - i\tilde{b}_{n+1,k}(\sigma_{n+1},\sigma_{k'}), \quad k \neq p,\]
\[(B_{\sigma})_{kp} = b_{kp}(\sigma_k,\sigma_p) - \tilde{b}_{kp}(\sigma_k,\sigma_p_1) - i\tilde{b}_{kp}(\sigma_k,\sigma_{p_1}),\]
\[(B_{\sigma})_{kp} = b_{kp}(\sigma_k,\sigma_p) - \tilde{b}_{kp}(\sigma_k,\sigma_{p_1}) + i\tilde{b}_{kp}(\sigma_k,\sigma_{p_1}),\]
\[(B_{\sigma})_{kp'} = b_{kp}(\sigma_{k'},\sigma_p) - \tilde{b}_{kp}(\sigma_{k'},\sigma_{p_1}) - i\tilde{b}_{kp}(\sigma_{k'},\sigma_{p_1}),\]
\[(B_{\sigma})_{kp'} = b_{kp}(\sigma_{k'},\sigma_p) - \tilde{b}_{kp}(\sigma_{k'},\sigma_{p_1}) + i\tilde{b}_{kp}(\sigma_{k'},\sigma_{p_1}).\]

Here $b, b', \tilde{b}, \tilde{b}' \in \mathbb{C}$ are expressed by the matrix elements of $A$ with the formula

\[b_{n+1,n+1} = a_{\sigma_{n+1},\sigma_{n+1}},\]
\[b_{n+1,k} = \frac{1}{\sqrt{2}} a_{\sigma_{n+1},\sigma_k}, \quad b_{k,n+1} = \frac{1}{\sqrt{2}} a_{\sigma_k,\sigma_{n+1}},\]
\[\tilde{b}_{n+1,k} = \frac{1}{\sqrt{2}} a_{\sigma_{n+1},\sigma'_{k'}}, \quad \tilde{b}_{k,n+1} = \frac{1}{\sqrt{2}} a_{\sigma_k,\sigma'_{n+1}},\]
\[b_{kk} = \frac{1}{2} (a_{\sigma_k \sigma_k} + a_{\sigma_{k'} \sigma_{k'}}), \quad \tilde{b}_{kk} = \frac{1}{2} (a_{\sigma_k \sigma'_{k'}} - a_{\sigma_{k'} \sigma_k}),\]
\[b_{k'k} = \frac{1}{2} (a_{\sigma_k \sigma_{k'}} - a_{\sigma_{k'} \sigma_k}), \quad \tilde{b}_{k'k} = \frac{1}{2} (a_{\sigma_k \sigma_{k'}} + a_{\sigma_{k'} \sigma_k}),\]
\[b_{kp} = \frac{1}{2} a_{\sigma_k \sigma_p}, \quad b_{kp}' = \frac{1}{2} a_{\sigma_{k'} \sigma_{p'}}, \quad \tilde{b}_{kp} = \frac{1}{2} a_{\sigma_k \sigma_p}, \quad \tilde{b}_{kp}' = \frac{1}{2} a_{\sigma_{k'} \sigma_{p'}}, \quad k \neq p,\]

Let us observe that the elements $b$ of $B_{\sigma}(j)$ are obtained from the elements $b^*$ of $B_{\sigma}(j = 1)$ by multiplications on some products of parameters $j$, namely

\[b_{n+1,n+1}^* = b_{n+1,n+1},\quad b_{kk}^* = b_{kk},\quad b_{k'k}^* = b_{k'k},\]
\[b_{k,n+1}^* = (\sigma_k, \sigma_{n+1})b_{k,n+1},\quad b_{n+1,k}^* = (\sigma_k, \sigma_{n+1})b_{n+1,k},\]
\[b_{k',n+1}^* = (\sigma_{k'}, \sigma_{n+1})b_{k',n+1},\quad b_{n+1,k}' = (\sigma_{k'}, \sigma_{n+1})b_{n+1,k},\]
\[b_{kp}^* = (\sigma_k, \sigma_p)b_{kp},\quad b_{kp}' = (\sigma_k', \sigma_p')b_{kp},\]
\[b_{kp}^* = (\sigma_k, \sigma_{p'})b_{kp},\quad b_{kp}' = (\sigma_{k'}, \sigma_p)b_{kp}, \quad k \neq p.\]

A transformation of group by multiplications of some or all its group parameters on zero tending parameter $\epsilon$ is named as group contraction [1], if a
new group is obtained in the limit. The formulas (6) are just an example of such transformation, where the nilpotent values \( j_k = \iota_k \) are used instead of the limit \( \epsilon \to 0 \). In other words group contractions may be described mathematically correctly by the replacement of real or complex group parameters with a new one’s which are elements of Pimenov algebra \( D_n(\iota; \mathbb{C}) \). In our case such replacement is made for matrix elements.

Let us consider as an example the group \( SO(3; j; \mathbb{C}) \). For identical permutation \( \sigma = (1, 2, 3) \) the matrix \( D_\sigma \) is given by equation (3) for \( N = 3 \) and in symplectic basis the group \( SO(3; j; \mathbb{C}) \) is described by the matrices

\[
B_\sigma(j) = \begin{pmatrix}
    b_{11} + ij_1 j_2 \tilde{b}_{11} & j_1 b_{12} - ij_2 \tilde{b}_{12} & b_{31} - ij_1 j_2 \tilde{b}_{31} \\
    j_1 b_{21} + ij_2 \tilde{b}_{21} & b_{22} & j_1 b_{21} - ij_2 \tilde{b}_{21} \\
    b_{31} + ij_1 j_2 \tilde{b}_{31} & j_1 b_{12} + ij_2 \tilde{b}_{12} & b_{11} - ij_1 j_2 \tilde{b}_{11}
\end{pmatrix}.
\]

For \( \sigma = (2, 1, 3) \) one obtain from equation (4)

\[
B_\sigma(j) = \begin{pmatrix}
    b_{11} + i j_2 \tilde{b}_{11} & j_1 b_{12} - ij_1 j_2 \tilde{b}_{12} & b_{31} - i j_2 \tilde{b}_{31} \\
    j_1 b_{21} + ij_1 j_2 \tilde{b}_{21} & b_{22} & j_1 b_{21} - ij_1 j_2 \tilde{b}_{21} \\
    b_{31} + i j_2 \tilde{b}_{31} & j_1 b_{12} + ij_1 j_2 \tilde{b}_{12} & b_{11} - i j_2 \tilde{b}_{11}
\end{pmatrix},
\]

finally the permutation \( \sigma = (1, 3, 2) \) leads to the matrices

\[
B_\sigma(j) = \begin{pmatrix}
    b_{11} + i j_1 \tilde{b}_{11} & j_1 j_2 b_{12} - i j_2 \tilde{b}_{12} & b_{31} - i j_1 \tilde{b}_{31} \\
    j_1 j_2 b_{21} + i j_2 \tilde{b}_{21} & b_{22} & j_1 j_2 b_{21} - i j_2 \tilde{b}_{21} \\
    b_{31} + i j_1 \tilde{b}_{31} & j_1 j_2 b_{12} + i j_2 \tilde{b}_{12} & b_{11} - i j_1 \tilde{b}_{11}
\end{pmatrix}.
\]

The same matrices are corresponded to three remaining permutations from the group \( S(3) \).

For nilpotent values of both parameters \( j_1 = \iota_1, j_2 = \iota_2 \) we have the complex Galilei group \( G(1 + 1; \mathbb{C}) = SO(3; \iota; \mathbb{C}) \), which is realized in Cartesian basis by the matrices

\[
A(\iota) = \begin{pmatrix}
    1 & \iota_1 a_{12} & \iota_1 \iota_2 a_{13} \\
    -\iota_1 a_{12} & 1 & \iota_2 a_{23} \\
    \iota_1 \iota_2 a_{31} & -\iota_2 a_{23} & 1
\end{pmatrix},
\]

where \( a_{31} = -a_{13} + a_{12} a_{23} \). The relations of \( j \)-orthogonality (1) have been taken into account. Three different realizations of Galilei group in symplectic description are as follows

\[
B_\sigma(\iota) = \begin{pmatrix}
    1 + i \iota_1 \iota_2 \tilde{b}_{11} & \iota_1 b_{12} - i \iota_2 \tilde{b}_{12} & -i \iota_1 \iota_2 \tilde{b}_{31} \\
    -\iota_1 b_{12} - i \iota_2 \tilde{b}_{12} & 1 & -\iota_1 b_{12} + i \iota_2 \tilde{b}_{12} \\
    i \iota_1 \iota_2 \tilde{b}_{31} & \iota_1 b_{12} + i \iota_2 \tilde{b}_{12} & 1 - i \iota_1 \iota_2 \tilde{b}_{11}
\end{pmatrix},
\]

7
where \( \tilde{b}_{31} = -b_{12} \tilde{b}_{12} \),

\[
B_{\sigma}(i) = \begin{pmatrix}
1 + i\nu_2 \tilde{b}_{11} & i\nu_1 \nu_2 \tilde{b}_{12} & 0 \\
-t_1 \nu_2 + i\nu_1 \nu_2 \tilde{b}_{21} & 1 & -t_1 \nu_1 \nu_2 \tilde{b}_{21} \\
0 & t_1 \nu_2 + i\nu_1 \nu_2 \tilde{b}_{12} & 1 - i\nu_2 \tilde{b}_{11}
\end{pmatrix},
\]

where \( \tilde{b}_{21} = -\tilde{b}_{12} - b_{12} \tilde{b}_{11} \),

\[
B_{\sigma}(i) = \begin{pmatrix}
1 + i\nu_1 \tilde{b}_{11} & i\nu_1 \nu_2 \tilde{b}_{12} & 0 \\
i\nu_1 \nu_2 \tilde{b}_{21} - i\nu_2 \tilde{b}_{12} & 1 & i\nu_1 \nu_2 \tilde{b}_{21} + i\nu_2 \tilde{b}_{12} \\
0 & i\nu_1 \nu_2 \tilde{b}_{12} + i\nu_2 \tilde{b}_{12} & 1 - i\nu_1 \tilde{b}_{11}
\end{pmatrix},
\]

where \( b_{21} = -b_{12} + \tilde{b}_{11} \tilde{b}_{12} \).

3 Constructions of quantum orthogonal groups.

3.1 Formal definition of quantum group \( SO_v(N; j; \sigma) \)

In the definition of the quantum group \( SO_v(N; j; \sigma) \) we shall follow [5], but start with an algebra \( D\langle (T_{\sigma})_{ik} \rangle \) of noncommutative polynomials of \( N^2 \) variables, which are elements of the direct product \( D_{N-1}(j) \otimes \mathbb{C}\langle t_{ik} \rangle \). More precisely, the elements \( (T_{\sigma})_{ik} \) are obtained from the elements \( (B_{\sigma}(j))_{ik} \) of equations (5) by the replacement of commutative variables \( b, b', \tilde{b}, \tilde{b}' \) with the noncommutative variables \( t, t', \tau, \tau' \), respectively. It is clear that generators \( t, t', \tau, \tau' \) are connected with the corresponding generators \( t^*, t'^*, \tau^*, \tau'^* \) of \( SO_q(N) \) in just the same way (6) as elements \( b, b', \tilde{b}, \tilde{b}' \) are connected with \( b^*, b'^*, \tilde{b}^*, \tilde{b}'^* \). One introduces additionally the transformation of the deformation parameters \( q = e^z \) as follows:

\[
z = Jv, \tag{7}
\]

where \( v \) is a new deformation parameter and \( J \) is some product of parameters \( j \) for the present unknown. Nondegenerate low triangular matrix \( R_q \in M_{N^2}(\mathbb{C}) \) is given by

\[
R_q = q \sum_{k=1, k \neq k'}^N e_{kk} \otimes e_{kk} +
\]

\[
+ \sum_{k, r=1, k \neq r, r'} e_{kk} \otimes e_{rr} + q^{-1} \sum_{k, k \neq k'} e_{k'k} \otimes e_{kk} + (q^{-1}) \sum_{k, r=1, k < r} e_{kr} \otimes e_{rk} -
\]

8
\[-(q - q^{-1}) \sum_{k,r=1, k>r}^{N} q^{p_k - p_r} e_{kr} \otimes e_{k'r'} + e_{pp} \otimes e_{pp},\]

where the last term is present only for \( N = 2n + 1 \) and \( p = (N + 1)/2 \). Here \( e_{ik} \in M_N(\mathbb{C}) \) are the matrix units \((e_{ik})_{sm} = \delta_{is}\delta_{km}, k' = N + 1 - k, \ r' = N + 1 - r \) and

\[(\rho_1, \ldots, \rho_N) = \begin{cases} 
(n - \frac{1}{2}, n - \frac{3}{2}, \ldots, \frac{1}{2}, 0, -\frac{1}{2}, \ldots, -n + \frac{1}{2}), & N = 2n + 1 \\
(n - 1, n - 2, \ldots, 1, 0, 0, -1, \ldots, -n + 1), & N = 2n.
\end{cases}\] (8)

Matrix \( C \) is as follows

\[C = C_0 q^\rho, \ \rho = \text{diag}(\rho_1, \ldots, \rho_N), \ (C_0)_{ik} = \delta_{i'k}, \ i, k = 1, \ldots, N,\]

\[(C)_{ik} = q^{p_i'} \delta_{i'k}, \ (C^{-1})_{ik} = q^{-p_i} \delta_{i'k}.\]

Let \( R_v(j), C(j) \) be matrices which are obtained from \( R_q, C \) by the replacement of deformation parameter \( z \) with \( Jv \):

\[R_v(j) = R_q(z \rightarrow Jv), \quad C(j) = C(z \rightarrow Jv).\]

The commutation relations of the generators \( T_\sigma(j) \) are defined by

\[R_v(j)T_1(j)T_2(j) = T_2(j)T_1(j)R_v(j),\] (9)

where \( T_1(j) = T_\sigma(j) \otimes I, \ T_2(j) = I \otimes T_\sigma(j) \) and the additional relations of \((v, j)\)-orthogonality

\[T_\sigma(j)C(j)T_\sigma^t(j) = T_\sigma^t(j)C(j)T_\sigma(j) = C(j).\] (10)

are imposed.

One defines the quantum orthogonal Cayley-Klein group \( SO_v(N; j; \sigma) \) as the quotient algebra of \( \mathbf{D}(\langle T_\sigma \rangle_{ik}) \) by relations (9),(10). Formally \( SO_v(N; j; \sigma) \) is a Hopf algebra with the following coproduct \( \Delta \), counit \( \epsilon \) and antipode \( S \):

\[\Delta T_\sigma(j) = T_\sigma(j) \otimes T_\sigma(j), \quad \epsilon(T_\sigma(j)) = I, \quad S(T_\sigma(j)) = C(j)T_\sigma^t(j)C^{-1}(j).\]

In terms of generators \( t, \tau \) the explicit form of antipode is given in Appendix A, of coproduct is given in Appendix B and of \((v, j)\)-orthogonality relations are given in Appendix C. As far as only secondary diagonal elements of the matrix \( C(j) \) are different from zero and for \( q = 1, j = 1 \) it is equal to \( C_0 \), then we have the symplectic description of \( SO_v(N; j; \sigma) \).
3.2 Allowed contractions of $SO_v(N; j; \sigma)$

The formal definition of the quantum group $SO_v(N; j; \sigma)$ should be a real definition of quantum group, if the proposed construction is a consistent Hopf algebra structure under nilpotent values of some or all parameters $j$. Counit $\epsilon(t_{n+1,n+1}) = 1$, $\epsilon(t_{kk}) = 1$, $k = 1, \ldots, n$, and $\epsilon(t) = \epsilon(\tau) = 0$ for the rest generators do not restrict the values of parameters $j$. Parameters $j$ are arranged in the expressions for coproduct $\Delta$ (Appendix B) exactly as in matrix product of $B_\sigma(j)$, and as far as the last ones form the group $SO(N; j; \mathbb{C})$ for any values of $j$, no restrictions follow from the coproduct. Different situation is with the antipode $S$ (Appendix A). Really, for elements

\[(T_{\sigma})_{k'k} = t_{k'k} + i\tau_{k'k}(\sigma_k, \sigma_{k'}), \quad k = 1, \ldots, n, \tag{11}\]

antipode is obtained as

\[S((T_{\sigma})_{k'k}) = (T_{\sigma})_{k'k} \cdot e^{2J\rho_k v}, \tag{12}\]

and depend both on $\rho_k$ and for the present undetermined factor $J$. Antipode is antihomomorphism of Hopf algebra and therefore has to transform the matrix $T_{\sigma}(j)$ to a matrix with the same distribution of the nilpotent parameters $j$ in its elements, i.e. the right and the left parts of equation (12) must be identical elements of $D_{N-1}(j) \otimes \mathbb{C}(t_{ik})$. For $J = 1$ this condition holds for any values of the parameters $j$. The case $J \neq 1$ requires additional discussion.

Next condition which must be taken into account is the $(v, j)$-orthogonality relations (10) (see Appendix C). In general, for nilpotent values of parameters $j_k$ the number of equations (10) are increased as compared with the case $j_k = 1$, $k = 1, \ldots, n$ because it is necessary equate to each other terms with nilpotent generators and their products independently. Then the number of contracted quantum group generators are decreased as compared with the initial $SO_q(N)$. For example, for $j_k = t_k$, $k = 1, \ldots, n$, $J = 1$ the only nonzero generators of quantum group $SO_q(2n+1; t; \sigma_k = k)$ are $\tau_{kk}, \tau_{k'k}, t_{kk} = 1$, $k = 1, \ldots, n$, $t_{n+1,n+1} = 1$, i.e. as a result of contraction we have the Hopf algebra with the number of generators equal to $2n - 1$ which is less then $N(N - 1)/2 = n(2n + 1)$.

On the other hand most interesting are such contractions, when the number of generators is conserved. It is necessary for this that the number of equations in $(v, j)$-orthogonality relations is not changed as compared with the initial quantum group. It is possible when nilpotent generators appear in equation (10) either with the powers greater or equal two (and then
the corresponding terms are equal to zero) or as homogeneous multipliers.
Taking into account all these arguments and using the explicit expressions for antipode and \((v, j)\)-orthogonality we can find possible contractions of quantum orthogonal groups, which are described by the following theorems.

**Theorem 1.** If the deformation parameter is not transformed \(J = 1\), then the following maximal \(n\)-dimensional contraction of the orthogonal quantum group \(SO_v(N; j; \sigma)\), \(N = 2n + 1\) is allowed:

\[
j_{2s} = v_{2s}, \quad s = 1, \ldots, m, \quad j_{2r+1} = v_{2r+1}, \quad r = m, \ldots, n-1, \quad 0 \leq m \leq n, \quad (13)
\]

for example, for permutation \(\sigma\): \(\sigma_{n+1} = 2m + 1, \quad \sigma_s = 2s - 1, \quad \sigma_{s'} = 2s, \quad s = 1, \ldots, m, \quad \sigma_r = 2r, \quad \sigma_{r'} = 2r + 1, \quad r = m + 1, \ldots, n.

**Proof.** From the explicit form of \((v, j)\)-orthogonality (Appendix C) it follows that if all multipliers \((\sigma_k, \sigma_{k'})\), \(k = 1, \ldots, n\) are equal to one (or equivalently \(\bigcup_{k=1}^{n} (\sigma_k, \sigma_{k'}) = 1\)), then under conditions of theorem all products of the parameters \(j\) in (10) are equal to zero, otherwise are appeared in these equations as homogeneous multipliers.

**Theorem 2.** If the deformation parameter is not transformed \(J = 1\), then the following maximal \(n\)-dimensional contraction of the quantum orthogonal group \(SO_v(N; j; \sigma)\), \(N = 2n\) is allowed:

\[
j_{2s} = v_{2s}, \quad s = 1, \ldots, m - 1, \quad j_{2p-1} = v_{2p-1}, \quad p = m, \ldots, u, \\
j_{2r} = v_{2r}, \quad r = u, \ldots, n - 1, \quad 1 \leq m \leq u \leq n, \quad (14)
\]

for example, for permutation \(\sigma\): \(\sigma_n = 2m - 1, \quad \sigma_{n'} = 2u, \quad \sigma_s = 2s - 1, \quad \sigma_{s'} = 2s, \quad s = 1, \ldots, m - 1, \quad \sigma_p = 2p, \quad \sigma_{p'} = 2p + 1, \quad p = m, \ldots, u - 1, \quad \sigma_r = 2r + 1, \quad \sigma_{r'} = 2r, \quad r = u, \ldots, n - 1.

**Proof.** Similar to the proof theorem 1, except \(k = 1, \ldots, n - 1\).

**Remark 1.** It should be noted that as \(\sigma\) may be taken any permutation with the properties \((\sigma_k, \sigma_{k'}) = 1, \quad k = 1, \ldots, n\) (or \(n - 1\)).

**Remark 2.** Admissible contractions for number of parameters \(j_k\) less then \(n\) are obtained from (13), (14), by setting part of \(j_{2s}, j_{2p-1}, j_{2r}, j_{2r+1}\) equal to one.

We return to the antipode (12) for \(J \neq 1\). As far as \(\rho_{n+1} = 0\) for \(N = 2n + 1\), and \(\rho_n = \rho_{n'} = 0\) for \(N = 2n\), (8) we shall regard these two cases separately.

**Theorem 3.** If the deformation parameter is transformed \((J \neq 1)\), then the following contractions of the quantum orthogonal group \(SO_v(N; j; \sigma)\), \(N = 2n + 1\) are allowed:

1. \(J = j_{n+1},\)
\[ j_{n+1} = \iota_{n+1}, \quad \text{if } 1 < \sigma_{n+1} < n + 1; \]
\[ j_n = \iota_n, \quad \text{if } n + 1 < \sigma_{n+1} < 2n + 1; \]
\[ j_n = \iota_n, \quad \text{if } \sigma_{n+1} = 1. \]

2. For \( J = j_n \),
   a) \( j_n = \iota_n \), \quad if \( 1 < \sigma_{n+1} < n + 1; \)
   b) \( j_n = \iota_n, \quad j_2n = 1, \iota_{2n} \), \quad if \( \sigma_{n+1} = 2n + 1. \)

3. For \( J = j_n j_{n+1} \),
   \[ j_n = 1, \iota_n, \quad j_{n+1} = 1, \iota_{n+1}, \quad \text{if } \sigma_{n+1} = n + 1. \]

**Proof.** If \( J \sim \iota \), then \( e^{J_{pw}} = 1 + J_{pw} \), and equation (12) is rewritten as
\[
S(T_{k'k}) = t_{k'k} + i\tau_{k'k}(\sigma_k, \sigma_{k'}) + 2t_{k'k}\rho_k v J + 2i\tau_{k'k}(\sigma_k, \sigma_{k'})\rho_k v J. \quad (15)
\]

The terms with factor \( J \) may be added only with the terms with factors \((\sigma_k, \sigma_{k'})\), \( k = 1, \ldots, n \), therefore
\[
J = \bigcap_{k=1}^{n} (\sigma_k, \sigma_{k'}), \quad (16)
\]
i.e. multiplier \( J \) is the product of all nilpotent generators of Pimenov algebra, which are simultaneously contained in all \((\sigma_k, \sigma_{k'})\), \( k = 1, \ldots, n \). Intersection is not empty if all \( \sigma_k \) are less then all \( \sigma_{p'} \), i.e. \( \sigma_k < \sigma_{p'} \), \( \forall k, p = 1, \ldots, n \).

The terms \( J(\sigma_k, \sigma_{k'}) \) are equal to zero. Taking into account that \( \sigma_{n+1} \) is not contained in equation (16) we may find unknown multiplier \( J \).

A. If \( \sigma_{n+1} < n + 1 \), then \( \max \sigma_p = n + 1 \) and \( \min \sigma_{p'} = n + 2 \), therefore
\[ J = \bigcap_{p=1}^{n} (\sigma_k, \sigma_{k'}) = (n + 1, n + 2) = j_{n+1}. \]

B. If \( \sigma_{n+1} > n + 1 \), then \( \max \sigma_p = n \) and \( \min \sigma_{p'} = n + 1 \), therefore
\[ J = \bigcap_{p=1}^{n} (\sigma_k, \sigma_{k'}) = (n, n + 1) = j_n. \]

C. If \( \sigma_{n+1} = n + 1 \), then \( \max \sigma_p = n \) and \( \min \sigma_{p'} = n + 2 \), therefore
\[ J = \bigcap_{p=1}^{n} (\sigma_k, \sigma_{k'}) = (n, n + 2) = j_n j_{n+1}. \]

Let us return to (15) with regard of obtained possible values \( J \).

1. For \( J = j_{n+1} \).
   a) For permutations \( \sigma \) with \( 1 < \sigma_{n+1} < n + 1 \) only one contraction \( j_{n+1} = \iota_{n+1} \) is allowed.
   b) If \( \sigma_{n+1} = 1 \), then all products \((\sigma_k, \sigma_{k'}), k = 1, \ldots, n \) do not contain parameter \( j_1 \), therefore it is not appeared in \((T_{\sigma})_{kk'} \), \( S((T_{\sigma})_{kk'}). \) Consequently for nilpotent value of \( j_1 \) the above mentioned matrix elements and their antipodes are the same elements of \( D_{N-1}(j) \otimes C \langle \iota_{kk} \rangle \), i.e. for permutations \( \sigma \) with \( \sigma_{n+1} = 1 \) two dimensional contraction \( j_{n+1} = \iota_{n+1}, j_1 = \iota_1 \) is allowed.

2. For \( J = j_n \).
   a) For permutations \( \sigma \) with \( n + 1 < \sigma_{n+1} < 2n + 1 \) only one contraction \( j_n = \iota_n \) is allowed.
b) For permutations $\sigma$ with $\sigma_{n+1} = 2n + 1$ two dimensional contraction $j_n = \ell_n$, $j_{2n} = \ell_{2n}$ is allowed since all products $(\sigma_k, \sigma_{k'})$, $k = 1, \ldots, n$ do not contain parameters $j_{2n}$.

3. For $J = j_n j_{n+1}$ the permutations $\sigma$ with $\sigma_{n+1} = n+1$ are regarded and both parameters $j_n, j_{n+1}$ may be independently equal to nilpotent values, therefore one have three contractions: $j_n = \ell_n$, $j_{n+1} = \ell_{n+1}$ and $j_{n+1} = \ell_{n+1}, j_{n+1} = \ell_{n+1}$.

We have found the admissible contractions by analysis of antipode of the matrix elements $(T_n)_{k'k}$, $k = 1, \ldots, n$. One may verify that the antipode of remaining elements of $T_n$ leads to the same admissible contractions. In this sense the selected elements $(T_\sigma)_{k'k}$ are most informative.

Using the explicit form of $(v, j)$-orthogonality (Appendix C), it is easy to verify that under the conditions of theorem all products of the parameters $j$ are equal to one or zero, otherwise are appeared in (10) as homogeneous multipliers.

**Theorem 4.** If the deformation parameter is transformed ($J \neq 1$), then the following contractions of the quantum orthogonal group $SO_q(N; j; \sigma)$, $N = 2n$ are allowed:

1. For $J = j_n$,
   a) $j_n = \ell_n$, if $\sigma_n > 1$, $\sigma_{n'} < 2n$;
   b) $j_n = \ell_n$, $j_1 = 1, \ell_1$, if $\sigma_n = 1$, $\sigma_{n'} < 2n$;
   c) $j_n = \ell_n$, $j_{2n-1} = 1, \ell_{2n-1}$, if $\sigma_n > 1$, $\sigma_{n'} = 2n$;
   d) $j_n = \ell_n$, $j_1 = 1, \ell_1$, $j_{2n-1} = 1, \ell_{2n-1}$, if $\sigma_n = 1$, $\sigma_{n'} = 2n$.

2. For $J = j_{n-1}$.
   a) $j_{n-1} = \ell_{n-1}$, if $\sigma_{n'} < 2n$;
   b) $j_{n-1} = \ell_{n-1}$, $j_2 = 1, \ell_2$, if $\sigma_n < 2n - 1$, $\sigma_{n'} = 2n$;
   c) $j_{n-1} = \ell_{n-1}$, $j_{2n-2} = 1, \ell_{2n-2}$, if $\sigma_n = 2n - 1$, $\sigma_{n'} = 2n$.

3. For $J = j_{n+1}$.
   a) $j_{n+1} = \ell_{n+1}$, if $\sigma_n > 1$;
   b) $j_{n+1} = \ell_{n+1}$, $j_1 = 1, \ell_1$, if $\sigma_n = 1$, $\sigma_{n'} > 2$;
   c) $j_{n+1} = \ell_{n+1}$, $j_1 = 1, \ell_1$, $j_2 = 1, \ell_2$, if $\sigma_n = 1$, $\sigma_{n'} = 2$.

4. For $J = j_{n-1} j_n$.
   a) $j_{n-1} = \ell_{n-1}$, $j_n = \ell_n$, if $\sigma_{n'} < 2n$;
   b) $j_{n-1} = \ell_{n-1}$, $j_n = \ell_n$, $j_{2n-1} = 1, \ell_{2n-1}$, if $\sigma_{n'} = 2n$.

5. For $J = j_n j_{n+1}$.
   a) $j_n = \ell_n$, $j_{n+1} = \ell_{n+1}$, if $\sigma_n > 1$;
   b) $j_1 = 1, \ell_1$, $j_n = \ell_n$, $j_{n+1} = \ell_{n+1}$, if $\sigma_n = 1$.

6. For $J = j_{n-1} j_n j_{n+1}$.
   a) $j_{n-1} = 1, \ell_{n-1}$, $j_n = 1, \ell_n$, $j_{n+1} = 1, \ell_{n+1}$, if $\sigma_n = n$, $\sigma_{n'} = n + 1$. 

13
The matrix elements \((T_{\sigma})_{k', k}\) and their antipodes are described by equations (11),(12) with \(k = 1, \ldots, n - 1\). (As far as \(\rho_n = 0\), then \(S((T_{\sigma})_{n,n'}) = S((T_{\sigma})_{n,n+1}) = (T_{\sigma})_{n,n+1}\)). Therefore \(J\) is given by equations (16) with the replacement of \(n\) by \(n-1\). Considering \(\sigma_k < \sigma_{p'}\), \(\forall k, p = 1, \ldots, n\), one finds admissible values of \(J\).

A. Let \(\sigma_n < n\) and \(\sigma_{n'} > n + 1\), then \(\max \sigma_k = n\) and \(\min \sigma_{p'} = n + 1\), hence \(J = (n, n + 1) = j_n\).

B. Let \(\sigma_{n'} > \sigma_n > n\), then \(\max \sigma_p = n - 1\) and \(\min \sigma_{p'} = n\), hence \(J = (n - 1, n) = j_{n-1}\).

C. Let \(n \geq \sigma_{n'} > \sigma_n\), then \(\max \sigma_p = n + 1\) and \(\min \sigma_{p'} = n + 2\), hence \(J = (n + 1, n + 2) = j_{n+1}\).

D. Let \(\sigma_n = n, \sigma_{n'} > n + 1\), then \(\max \sigma_p = n - 1\) and \(\min \sigma_{p'} = n + 1\), hence \(J = (n - 1, n + 1) = j_{n-1}j_n\).

E. Let \(\sigma_n < n, \sigma_{n'} = n + 1\), then \(\max \sigma_p = n\) and \(\min \sigma_{p'} = n + 2\), hence \(J = (n, n + 2) = j_nj_{n+1}\).

F. Let \(\sigma_n = n, \sigma_{n'} = n + 1\), then \(\max \sigma_p = n - 1\) and \(\min \sigma_{p'} = n + 2\), hence \(J = (n - 1, n + 2) = j_{n-1}j_nj_{n+1}\).

The analysis of equations (12), with due regard for obtained possible values of \(J\), leads to the admissible contractions of the theorem. Using the explicit form of \((v, j)\)-orthogonality (Appendix C), it is easy to verify that under the conditions of theorem all products of the parameters \(j\) are equal to one or zero, otherwise are appeared in (10) as homogeneous multipliers.

Hopf algebra \(SO_q(N; j; \sigma)\), \(N = 2n + 1\) has \(n\) primitive elements which correspond to \(n\) nonintersecting \(2 \times 2\) submatrices of the Cartesian matrix \(A(j)\) composed from elements \(a_{\sigma_k}, a_{\sigma_k\sigma_{k'}}, a_{\sigma_{k'}\sigma_k}, a_{\sigma_{k'}\sigma_{k'}}\), \(k = 1, \ldots, n\). Under the transition to the symplectic basis they are transformed to \(n\) diagonal \(2 \times 2\) submatrices \(\text{diag}(B_{\sigma_k}, (B_{\sigma_k})_{k', k}) = \text{diag}(b_{kk} + i\tilde{b}_{kk}(\sigma_k, \sigma_{k'}), b_{kk} - i\tilde{b}_{kk}(\sigma_k, \sigma_{k'}))\), \(k = 1, \ldots, n\), see (5). Each such matrix is either one parameter rotation subgroup \(SO(2)\), if \((\sigma_k, \sigma_{k'}) = 1\), or one parameter Galilei transformation \(SO(2; j = \iota) = G(1, 1)\), if \((\sigma_k, \sigma_{k'}) = \iota\). Therefore, if the deformation parameter \(\zeta\) is fixed \((J = 1)\), then \(n\) primitive elements of the contracted quantum orthogonal groups correspond to Euclidean rotation \(SO(2)\). If the deformation parameter is transformed \(z = \iota\), then \(n\) primitive elements correspond to Galilei transformation \(SO(2; j = \iota) = G(1, 1)\). The same is true for the contracted quantum groups \(SO_q(N; j; \sigma)\), \(N = 2n\).

Let us note that contractions of quantum orthogonal algebras with different sets of primitive elements have been discussed in [4],[24].

Quantum orthogonal groups have contractions with the same nilpotent parameters \(j\) both with a fixed deformation parameter and with a trans-
formed one. For example, the quantum group $SO_q(2n+1; j; \sigma)$ for even $n = 2p$ at $\sigma_{n+1} = 1$ according to (13) has contraction $j_n = \iota_n$, $j_{n+1} = \iota_{n+1}$, $J = 1$ and according to 3 of Theorem 3 has the same two-dimensional contraction, but $J = \iota_n \iota_{n+1}$. Quantum group $SO_q(2n; j; \sigma)$ for odd $n = 2p - 1$ at $\sigma_n = n$, $\sigma'_n = n + 1$ according to (14) has contraction $j_{n-1} = \iota_{n-1}$, $j_n = \iota_n$, $j_{n+1} = \iota_{n+1}$, $J = 1$ and according to 6 of Theorem 4 has the same three-dimensional contraction but $J = \iota_{n-1} \iota_n \iota_{n+1}$. Let us stress that the cases $J = 1$ and $J \sim \iota$ are realized for different sets of primitive elements in Hopf algebras $SO_q(4p+1; \iota_1 \iota_2 \iota_{n+1}; \iota)$ and $SO_q(4p-2; \iota_{n-1} \iota_n \iota_{n+1}; \iota)$, respectively.

Let permutation $\sigma$ be identical, i.e. $\sigma_k = k$, $\sigma_{k'} = k'$, $\sigma_{n+1} = n + 1$. It follows from theorems 1 and 2 that there are no contractions of quantum orthogonal group $SO_q(N; j)$ with fixed deformation parameter ($J = 1$). For $N = 2n + 1$ from theorem 3 we obtain three possible contractions $j_n = 1$, $\iota_n$, $j_{n+1} = 1$, $\iota_{n+1}$ (both parameters $j_n$ and $j_{n+1}$ independently take nilpotent values) and deformation parameters is transformed by (7), with $J = j_n j_{n+1}$. For $N = 2n$ from theorem 4 we obtain seven admissible contractions: $j_{n-1} = 1$, $\iota_{n-1}$, $j_n = 1$, $\iota_n$, $j_{n+1} = 1$, $\iota_{n+1}$, where deformation parameter is multiplied by $J = j_{n-1} j_n j_{n+1}$. It should be considered in papers [21],[25] just these allowed contractions.

4 Quantum complex kinematic groups

Kinematic groups are motion groups of the maximal homogeneous four-dimensional (one time and three space coordinates) space–time models [26]. All these groups may be obtained from the real orthogonal group $SO(5; \mathbb{R})$ by contractions and analytic continuations [22]. If one introduce Beltrami coordinates $\xi_k = x_{k+1}/x_1$, $k = 1, 2, 3, 4$ and one interpret $\xi_1$ as a time axis while the rest three – as a space axes, then Galilei group $G(1,3) = SO(5; \iota_1, \iota_2, 1, 1)$ is the motion group of the nonrelativistic space-time with zero curvature, Newton groups $N^\pm(1,3) = SO(5; j_1 = 1, i; \iota_2, 1, 1)$ are the motion groups of the nonrelativistic space-time with positive and negative curvature, respectively. Poincare group $P(1,3) = SO(5; \iota_1, i, 1, 1)$ is the motion group of the relativistic space-time with zero curvature and $S^\pm(1,3) = SO(5; j_1 = 1, i; i, 1, 1)$ are the motion groups of the anti de Sitter space-time (positive curvature) and de Sitter space-time (negative curvature).

If one interpret three first Beltrami coordinates as a space axes while the last one as a time axis, then the three exotic Carroll kinematics are ob-
tained, namely $C^0(1, 3) = SO(5; \iota_1, 1, 1, \iota_4)$, with zero curvature, $C^\pm(1, 3) = SO(5; j_1 = 1, i; 1, 1, \iota_4)$, with positive and negative curvature.

The groups $N^\pm(1, 3)$ are the real forms of the complex Newton group $N(4)$. Poincare group $P(1, 3)$ is the real form of the complex Euclid group $E(4)$, the groups $C^\pm(1, 3)$ are the real forms of the complex Carroll group $C(4)$. In this paper the quantum deformations of the complex orthogonal groups are regarded, therefore with the help of contractions of $SO_q(5)$ the quantum analogs of the complex kinematic groups may be obtained. Possible contractions of the complex quantum group $SO_q(5; j; \sigma)$ are described by the theorems 1,3 and are as follows: for $J = 1, j_1 = 1, \iota_1, j_3 = 1, \iota_3$ with $\sigma = (2,4,1,5,3)$; $j_2 = 1, \iota_2, j_3 = 1, \iota_3$ with $\sigma = (1,4,3,5,2)$; $j_2 = 1, \iota_2, j_4 = 1, \iota_4$ with $\sigma = (1,3,5,4,2)$; for $J = \iota_2, j_2 = \iota_2, j_4 = 1, \iota_4$ with $\sigma = (1,2,5,3,4)$; for $J = \iota_3, j_1 = 1, \iota_1, j_3 = \iota_3$ with $\sigma = (2,3,1,4,5)$; for $J = \iota_2\iota_3, j_2 = \iota_2, j_3 = \iota_3$ with $\sigma = (1,2,3,4,5)$. Thus if deformation parameter remains unchanged ($J = 1$), then we have the quantum analog of Euclidean group $E_q(4)$ for $j_1 = \iota_1, j_2 = j_3 = j_4 = 1, \sigma = (2,4,1,5,3)$; of Newton group $N_q(4)$ for $j_2 = \iota_2, j_1 = j_3 = j_4 = 1, \sigma = (1,4,3,5,2)$ and of Carroll group $C_q(4)$ for $j_4 = \iota_4, j_1 = j_2 = j_3 = 1, \sigma = (1,3,5,4,2)$. If deformation parameter is transformed under contraction $z = \iota_2 v$, then we have one more quantum deformation of Newton group $N_v(4)$ for $j_2 = \iota_2, j_1 = j_3 = j_4 = 1, \sigma = (1,2,5,3,4)$, which is not isomorphic to the previous one. Two primitive elements of $N_q(4)$ correspond to the elliptic translation along the time axis $t$ and to the rotation in the space plane $\{r_2, r_3\}$ (both are isomorphic to $SO(2)$), while primitive elements of $N_v(4)$ correspond to the flat translation along the spatial axis $r_2$ and to Galilei boost in the space-time plane $\{t, r_1\}$ (both are isomorphic to Galilei group $SO(2; j_2 = \iota_2) = G(1, 1)$). We did not obtain the quantum deformations of the complex Galilei $G(4)$ and Carroll $C^0(4)$ groups.

According to correspondence principle a new physical theory must include an old one as a particular case. For space-time theory this principle is realized as the chain of limit transitions: general relativity passes to special relativity, when space-time curvature tends to zero, and special relativity passes to classical physics, when light velocity tends to infinity. For kinematical groups this corresponds to the chain of contractions: $S^\pm(1, 3) \xrightarrow{K \rightarrow 0} P(1, 3) \xrightarrow{c \rightarrow \infty} G(1, 3)$. (17)

As it was mentioned above there is no quantum deformation of the complex Galilei group in our scheme, therefore we are not able to construct the standard quantum analog of the full chain of contractions (17), even at
the level of complex groups. This means that (at least standard) quantum deformation of the flat nonrelativistic (1+3) space-time does not exist in Cayley–Klein scheme.

5 Conclusion

From the contraction viewpoint Hopf algebra structure of quantum orthogonal group is more rigid as compared with a group one. Cayley-Klein groups are obtained [22] from $SO(N; j)$ for all nilpotent values of parameters $j_k, k = 1, \ldots, N-1$, whereas their quantum deformations exist only for some of them ($\leq \lfloor \frac{N}{2} \rfloor$). The main restrictions on contractions are appeared from antipode (12). In particular, contractions of quantum orthogonal groups with transformed deformation parameter $z = J\nu, J \neq 1$ are possible only due to some parameters (8), which characterize the matrix $R_q$, are equal to zero, namely $\rho_{n+1} = 0$ for $N = 2n + 1$ and $\rho_n = \rho_{n'} = 0$ for $N = 2n$. In this sense such contractions are exclusive and complementary to contractions with untransformed deformation parameter.

It should be noted that among the contracted for equal number of parameters $j$ quantum orthogonal groups may be isomorphic, as Hopf algebra quantum groups. Quantum groups isomorphism is not regarded in this paper.

Unlike of the undeformed case we are not able to obtain quantum deformation of Galilei group $G(1,3)$ by contraction of $SO_q(5)$. It seems that quantum groups and corresponding quantum spaces are not a suitable objects for simulation of noncommuting space-time because of the fundamental physical correspondence principle is not satisfied in this case.
A  Antipode  $S(T) = CT'C^{-1}$ of quantum group $SO_v(N, \sigma, j)$

\[
S(t_{n+1,n+1}) = t_{n+1,n+1}, \quad S(t_{kk}) = t_{kk}, \quad S(\tau_{kk}) = -\tau_{kk}, \\
S(t_{k',k}) = t_{k',k} \cosh 2J_{\rho_k}v + i\tau_{k',k}(\sigma_k, \sigma_{k'}) \sinh 2J_{\rho_k}v, \\
S(\tau_{k',k}) = \tau_{k',k} \cosh 2J_{\rho_k}v - it_{k',k}(\sigma_k, \sigma_{k'})^{-1} \sinh 2J_{\rho_k}v, \\
S(t_{n+1,k}) = t_{n+1,k} \cosh J_{\rho_k}v + it_{n+1,k} \frac{(\sigma_{k'}, \sigma_{n+1})}{(\sigma_k, \sigma_{n+1})} \sinh J_{\rho_k}v, \\
S(\tau_{n+1,k}) = \tau_{n+1,k} \cosh J_{\rho_k}v - it_{n+1,k} \frac{(\sigma_{k'}, \sigma_{n+1})}{(\sigma_k, \sigma_{n+1})} \sinh J_{\rho_k}v, \\
S(t_{k,n+1}) = t_{k,n+1} \cosh J_{\rho_k}v - it_{k,n+1} \frac{(\sigma_{k'}, \sigma_{n+1})}{(\sigma_k, \sigma_{n+1})} \sinh J_{\rho_k}v, \\
S(\tau_{k,n+1}) = \tau_{k,n+1} \cosh J_{\rho_k}v - it_{k,n+1} \frac{(\sigma_{k'}, \sigma_{n+1})}{(\sigma_k, \sigma_{n+1})} \sinh J_{\rho_k}v.
\]

\[
S(t_{pk}) = t_{pk} \cosh J_{\rho_k}v \sinh J_{\rho_p}v - \frac{(\sigma_{p'}, \sigma_{k'})}{(\sigma_k, \sigma_p)} t_{pk}' \cosh 2J_{\rho_k}v \sinh J_{\rho_p}v + \\
+i \frac{(\sigma_{p}, \sigma_{k'})}{(\sigma_k, \sigma_p)} \tau_{pk} \cosh J_{\rho_k}v \sinh J_{\rho_p}v + \\
+i \frac{(\sigma_{p'}, \sigma_k)}{(\sigma_k, \sigma_p')} t_{pk}' \cosh J_{\rho_k}v \sinh J_{\rho_p}v, \\
S(t'_{pk}) = t'_{pk} \cosh J_{\rho_k}v \sinh J_{\rho_p}v - t_{pk} \frac{(\sigma_{p}, \sigma_{k})}{(\sigma_{k'}, \sigma_{p'})} \cosh 2J_{\rho_k}v \sinh J_{\rho_p}v - \\
-i t_{pk} \frac{(\sigma_{p}, \sigma_{k'})}{(\sigma_{k'}, \sigma_{p'})} \cosh J_{\rho_k}v \sinh J_{\rho_p}v - \\
-i \frac{t_{pk}'}{(\sigma_{k'}, \sigma_{p'})} \sinh J_{\rho_k}v \sinh J_{\rho_p}v, \\
S(\tau_{kp}) = \tau_{kp} \cosh J_{\rho_k}v \sinh J_{\rho_p}v + \frac{(\sigma_{p}, \sigma_{k})}{(\sigma_{k'}, \sigma_{p'})} \cosh 2J_{\rho_k}v \sinh J_{\rho_p}v - \\
-it_{pk} \frac{(\sigma_{p}, \sigma_{k})}{(\sigma_{k}, \sigma_{p'})} \cosh J_{\rho_k}v \sinh J_{\rho_p}v + \\
+it_{pk}' \frac{(\sigma_{p'}, \sigma_{k'})}{(\sigma_{k}, \sigma_{p'})} \sinh J_{\rho_k}v \sinh J_{\rho_p}v, \\
S(\tau'_{kp}) = \tau'_{kp} \cosh J_{\rho_k}v \sinh J_{\rho_p}v - \frac{(\sigma_{p}, \sigma_{k'})}{(\sigma_{k'}, \sigma_{p'})} \cosh 2J_{\rho_k}v \sinh J_{\rho_p}v - \\
-it_{pk} \frac{(\sigma_{p}, \sigma_{k'})}{(\sigma_{k}, \sigma_{p'})} \cosh J_{\rho_k}v \sinh J_{\rho_p}v - \\
+it_{pk}' \frac{(\sigma_{p'}, \sigma_{k})}{(\sigma_{k}, \sigma_{p'})} \sinh J_{\rho_k}v \sinh J_{\rho_p}v.
\]
\[ S(\tau'_{kp}) = \tau_{pk} \cosh J\rho_k v \cosh J\rho_p v + \tau_{pk} \left( \frac{\sigma_{p'}, \sigma_k}{\sigma_{k'}, \sigma_p} \right) \sinh J\rho_k v \sinh J\rho_p v - \]
\[ -it_{pk} \left( \frac{\sigma_p, \sigma_k}{\sigma_{k'}, \sigma_p} \right) \sinh J\rho_k v \cosh J\rho_p v + \]
\[ +it'_{pk} \left( \frac{\sigma_{p'}, \sigma_{k'}}{\sigma_{k'}, \sigma_p} \right) \cosh J\rho_k v \sinh J\rho_p v. \]

**B Coproduct \( \Delta T = T \hat{\otimes} T \) of quantum group \( SO_v(N; \sigma; j) \)**

\[ \Delta_{n+1,n+1} = t_{n+1} \otimes t_{n+1} + 2 \sum_{k=1}^{n} \left[ (\sigma_k, \sigma_{n+1})^2 t_{n+1,k} \otimes t_{k,n+1} + (\sigma_{n+1}, \sigma_k)^2 \tau_{n+1,k} \otimes \tau_{k,n+1} \right], \]

\[ \Delta_{kk} = t_{kk} \otimes t_{kk} + t_{k'k} \otimes t_{k'k} + (\sigma_k, \sigma_{n+1})^2 t_{k,n+1} \otimes t_{n+1,k} + (\sigma_{n+1}, \sigma_k)^2 \tau_{k,n+1} \otimes \tau_{n+1,k} + (\sigma_k, \sigma_k)^2 (\tau_{k'k} \otimes \tau_{k'k} - \tau_{kk} \otimes \tau_{kk}) + 2 \sum_{s=1,s \neq k}^{n} \left[ (\sigma_k, \sigma_s)^2 t_{ks} \otimes t_{sk} + (\sigma_{k'}, \sigma_{s'})^2 t'_{ks} \otimes t'_{sk} + (\sigma_{s'}, \sigma_{s'})^2 \tau_{ks} \otimes \tau_{sk} + (\sigma_k, \sigma_{s'})^2 \tau_{ks} \otimes \tau_{sk} \right], \]

\[ \Delta_{r_{kk}} = \tau_{kk} \otimes \tau_{kk} + t_{kk} \otimes \tau_{kk} + t_{k'k} \otimes \tau_{k'k} \tau_{kk} + (\sigma_k, \sigma_{n+1})^2 (t_{k,n+1} \otimes \tau_{n+1,k} - \tau_{k,n+1} \otimes t_{n+1,k}) + \frac{2}{(\sigma_k, \sigma_{k'})} \sum_{s=1,s \neq k}^{n} \left[ (\sigma_k, \sigma_s)(\sigma_s, \sigma_{k'})(t_{ks} \otimes \tau_{sk} - t'_{ks} \otimes \tau_{sk}) + (\sigma_k, \sigma_{s})(\sigma_{s'}, \sigma_{k'})(\tau_{ks} \otimes t_{s}) - t'_{ks} \otimes \tau_{sk} \right], \]

\[ \Delta_{t_{k'k}} = t_{k'k} \otimes t_{k'k} + t_{kk} \otimes t_{k'k} + (\sigma_k, \sigma_{n+1})^2 t_{k,n+1} \otimes t_{n+1,k} - (\sigma_{n+1}, \sigma_k)^2 \tau_{n+1,k} \otimes \tau_{n+1,k} + (\sigma_k, \sigma_{k'})^2 (\tau_{kk} \otimes \tau_{k'k} - \tau_{k'k} \otimes \tau_{kk}) + 2 \sum_{s=1,s \neq k}^{n} \left[ (\sigma_k, \sigma_s)^2 t_{ks} \otimes t_{sk} - (\sigma_{k'}, \sigma_s)^2 t'_{ks} \otimes t'_{sk} + (\sigma_k, \sigma_{s})^2 \tau_{ks} \otimes \tau_{sk} - (\sigma_{k'}, \sigma_{s})^2 \tau_{ks} \otimes \tau_{sk} \right], \]

\[ \Delta_{r'_{k'k}} = \tau_{k'k} \otimes \tau_{k'k} + t_{kk} \otimes \tau_{k'k} + t_{k'k} \otimes \tau_{kk} - \tau_{kk} \otimes t_{k'k} + \]
\begin{align*}
\Delta t_{k,n+1} &= t_{k,n+1} \otimes t_{n+1,n+1} + (t_{kk} + t_{k'k}) \otimes t_{k,n+1} + \\
&\quad + \frac{2}{(\sigma_k,\sigma_{n+1})} \sum_{s=1,s \neq k}^{n} \left[ (\sigma_k,\sigma_s)(\sigma_s,\sigma_{n+1})t_{ks} \otimes t_{s,n+1} + \\
&\quad + (\sigma_k,\sigma_s')(\sigma_{s'},\sigma_{n+1})t_{ks} \otimes t_{s,n+1} \right],
\end{align*}

\begin{align*}
\Delta \tau_{k,n+1} &= \tau_{k,n+1} \otimes t_{n+1,n+1} + (t_{kk} - t_{k'k}) \otimes \tau_{k,n+1} + \\
&\quad + \frac{2}{(\sigma_{n+1},\sigma_{k'})} \sum_{s=1,s \neq k}^{n} \left[ (\sigma_{k'},\sigma_s)(\sigma_s,\sigma_{n+1})\tau_{ks} \otimes \tau_{s,n+1} + \\
&\quad + (\sigma_{k'},\sigma_{s'})(\sigma_{s'},\sigma_{n+1})\tau_{ks} \otimes \tau_{s,n+1} \right],
\end{align*}

\begin{align*}
\Delta t_{n+1,k} &= t_{n+1,n+1} \otimes t_{n+1,k} + t_{n+1,k} \otimes (t_{kk} + t_{k'k}) + \\
&\quad + \frac{2}{(\sigma_k,\sigma_{n+1})} \sum_{s=1,s \neq k}^{n} \left[ (\sigma_k,\sigma_s)(\sigma_s,\sigma_{n+1})t_{n+1,s} \otimes t_{sk} + \\
&\quad + (\sigma_k,\sigma_{s'})(\sigma_{s'},\sigma_{n+1})\tau_{n+1,s} \otimes \tau_{sk} \right],
\end{align*}

\begin{align*}
\Delta \tau_{n+1,k} &= t_{n+1,n+1} \otimes \tau_{n+1,k} + \tau_{n+1,k} \otimes (t_{kk} - t_{k'k}) + \\
&\quad + \frac{2}{(\sigma_{n+1},\sigma_{k'})} \sum_{s=1,s \neq k}^{n} \left[ (\sigma_{k'},\sigma_s)(\sigma_s,\sigma_{n+1})t_{n+1,s} \otimes \tau_{sk} + \\
&\quad + (\sigma_{k'},\sigma_{s'})(\sigma_{s'},\sigma_{n+1})\tau_{n+1,s} \otimes t_{sk} \right],
\end{align*}
\[
\Delta t_{kp} = t_{kp} \otimes (t_{pp} + t_{p'p}) + (t_{kk} + t_{kk'}) \otimes t_{kp} + \\
+ \frac{(\sigma_k, \sigma_{n+1})(\sigma_p, \sigma_{n+1})}{(\sigma_k, \sigma_p)} t_{k,n+1} \otimes t_{n+1,p} + \\
+ \frac{(\sigma_k, \sigma_{k'})(\sigma_p, \sigma_{k'})}{(\sigma_k, \sigma_p)} (\tau_{kk} + \tau_{k'k}) \otimes \tau_{kp}' + \\
+ \frac{(\sigma_k, \sigma_{p'})(\sigma_p, \sigma_{p'})}{(\sigma_k, \sigma_p)} \tau_{kp} \otimes (\tau_{pp} - \tau_{p'p}) + \\
+ \frac{2}{(\sigma_k, \sigma_p)} \sum_{s=1, s \neq k, p}^{n} \left[ (\sigma_k, \sigma_s)(\sigma_s, \sigma_p) t_{ks} \otimes t_{sp} + \\
+ (\sigma_k, \sigma_{s'})(\sigma_{s'}, \sigma_p) \tau_{ks} \otimes \tau_{sp}' \right],
\]

\[
\Delta t_{k'p} = t_{k'p} \otimes (t_{pp} - t_{p'p}) + (t_{kk} - t_{k'k}) \otimes t_{k'p} + \\
+ \frac{(\sigma_{n+1}, \sigma_{k'})(\sigma_{n+1}, \sigma_{p'})}{(\sigma_{k'}, \sigma_{p'})} t_{k,n+1} \otimes \tau_{n+1,p} + \\
+ \frac{(\sigma_k, \sigma_{k'})(\sigma_k, \sigma_{p'})}{(\sigma_{k'}, \sigma_{p'})} (\tau_{k'k} - \tau_{kk}) \otimes \tau_{kp} + \\
+ \frac{(\sigma_p, \sigma_{k'})(\sigma_p, \sigma_{p'})}{(\sigma_{k'}, \sigma_{p'})} \tau_{kp}' \otimes (\tau_{pp} + \tau_{p'p}) + \\
+ \frac{2}{(\sigma_{k'}, \sigma_{p'})} \sum_{s=1, s \neq k, p}^{n} \left[ (\sigma_{k'}, \sigma_{s'})(\sigma_{s'}, \sigma_p) t_{ks}' \otimes t_{sp}' + \\
+ (\sigma_{k'}, \sigma_s)(\sigma_s, \sigma_{p'}) \tau_{ks}' \otimes \tau_{sp} \right],
\]

\[
\Delta \tau_{kp} = \tau_{kp} \otimes (t_{pp} - t_{p'p}) + (t_{kk} + t_{k'k}) \otimes \tau_{kp} + \\
+ \frac{(\sigma_p, \sigma_{n+1})(\sigma_{n+1}, \sigma_{k'})}{(\sigma_k, \sigma_{p'})} \tau_{k,n+1} \otimes t_{n+1,p} + \\
+ \frac{(\sigma_k, \sigma_{k'})(\sigma_{k'}, \sigma_{p'})}{(\sigma_k, \sigma_{p'})} (\tau_{kk} + \tau_{k'k}) \otimes t_{kp}' + \\
+ \frac{(\sigma_k, \sigma_{p'})(\sigma_p, \sigma_{p'})}{(\sigma_k, \sigma_{p'})} \tau_{kp} \otimes (\tau_{pp} + \tau_{p'p}) + \\
+ \frac{2}{(\sigma_k, \sigma_{p'})} \sum_{s=1, s \neq k, p}^{n} \left[ (\sigma_k, \sigma_{s'})(\sigma_{s'}, \sigma_{p'}) \tau_{ks} \otimes t_{sp}' + \\
+ (\sigma_k, \sigma_s)(\sigma_s, \sigma_{p'}) t_{ks} \otimes \tau_{sp} \right],
\]
\[ \Delta \tau'_{kp} = \tau'_{kp} \otimes (t_{pp} + t_{p'p}) + (t_{kk} - t_{k'k}) \otimes \tau'_{kp} + \\
+ \frac{(\sigma_k, \sigma_{n+1})(\sigma_{n+1}, \sigma_p)}{(\sigma_k', \sigma_p)} t_{k,n+1} \otimes \tau_{n+1,p} + \\
+ \frac{(\sigma_k, \sigma_k')(\sigma_k, \sigma_p)}{(\sigma_k', \sigma_p)} (\tau_{k'k} - \tau_{kk}) \otimes t_{kp} + \\
+ \frac{(\sigma_k', \sigma_{p'})(\sigma_p, \sigma_{p'})}{(\sigma_k', \sigma_p)} t_{kp} \otimes (\tau_{p'p} - \tau_{pp}) + \\
+ \frac{2}{(\sigma_k', \sigma_p)} \sum_{s=1, s \neq k,p}^{n} (\sigma_{k'}, \sigma_s)(\sigma_s, \sigma_p) \tau'_{ks} \otimes t_{sp} + \\
+ (\sigma_{k'}, \sigma_{s'})(\sigma_{s'}, \sigma_p) t_{ks} \otimes \tau'_{sp}. \]

\[ C \quad (q - j)\text{-orthogonality relations } TCT'^{t} = C \text{ for quantum group } SO_v(N; \sigma; j) \]

Let us introduce the notation \( v_k = \rho_k v \).

\[ 1 = \tau^2_{n+1,n+1} + 2 \sum_{p=1}^{n} \left\{ (\sigma_p, \sigma_{n+1})^2 \tau^2_{n+1,p} + (\sigma_{n+1}, \sigma_{p'})^2 \tau^2_{n+1,p} \right\} \cosh Jv_p + \\
i(\sigma_p, \sigma_{n+1})(\sigma_{n+1}, \sigma_{p'})[t_{n+1,p}, \tau_{n+1,p}] \sinh Jv_p, \]

\[ \frac{1}{2} \cosh Jv_k = (\sigma_k, \sigma_{n+1})^2 \tau^2_{k,n+1} + 2 \sum_{p=1, p \neq k}^{n} \left\{ (\sigma_k, \sigma_p)^2 \tau^2_{kp} + \\
+ (\sigma_k, \sigma_{p'})^2 \tau^2_{kp} \right\} \cosh Jv_p + i(\sigma_k, \sigma_p)(\sigma_k, \sigma_{p'})[t_{kp}, \tau_{kp}] \sinh Jv_p \right\} + \\
+ \frac{1}{2} \left\{ t_{kk}^2 + t_{k'k}^2 + [t_{kk}, t_{k'k}]_+ + (\sigma_k, \sigma_{k'})^2 (\tau_{kk}^2 + \tau_{k'k}^2) + \\
+ [\tau_{kk}, \tau_{k'k}]_+ \right\} \cosh Jv_k + \frac{i}{2} (\sigma_k, \sigma_k') \left\{ [t_{kk}, \tau_{kk}] + \\
+ [t_{k'k}, \tau_{k'k}] + [t_{kk}, \tau_{k'k}] + [t_{k'k}, \tau_{kk}] \right\} \sinh Jv_k, \]

\[ \frac{1}{2} \cosh Jv_k = (\sigma_{n+1}, \sigma_{k'})^2 \tau^2_{k,n+1} + 2 \sum_{p=1, p \neq k}^{n} \left\{ (\sigma_{k'}, \sigma_p)^2 \tau^2_{kp} + \\
+ (\sigma_{k'}, \sigma_{p'})^2 \tau^2_{kp} \right\} \cosh Jv_p - i(\sigma_{k'}, \sigma_{p'})(\sigma_{k'}, \sigma_p)[t_{kp}, \tau_{kp}] \sinh Jv_p \right\} + \\
+ \frac{1}{2} \left\{ t_{kk}^2 + t_{k'k}^2 - [t_{kk}, t_{k'k}]_+ + (\sigma_k, \sigma_{k'})^2 (\tau_{kk}^2 + \tau_{k'k}^2) - \\
- [\tau_{kk}, \tau_{k'k}]_+ \right\} \cosh Jv_k + \frac{i}{2} (\sigma_{k'}, \sigma_{k'}) \left\{ [t_{kk}, \tau_{kk}] - \\
- [t_{k'k}, \tau_{k'k}] - [t_{kk}, \tau_{k'k}] - [t_{k'k}, \tau_{kk}] \right\} \sinh Jv_k, \]
\( \frac{1}{2} \sinh J v_k = -i(\sigma_k, \sigma_{n+1})(\sigma_{n+1}, \sigma_{k'}) t_{k,n+1} \tau_{k,n+1} + \\
+ 2 \sum_{p=1, p \neq k}^{n} \left\{ -i \left( (\sigma_k, \sigma_p)(\sigma_{k'}, \sigma_p)t_{kp} \tau_{kp} + \\
+ (\sigma_{k'}, \sigma_p)(\sigma_k, \sigma_p) \tau_{kp} t_{kp} \right) \cosh J v_p + \left( (\sigma_k, \sigma_p)(\sigma_{k'}, \sigma_p) t_{kp} t_{kp} - \\
- (\sigma_k, \sigma_p)(\sigma_{k'}, \sigma_{k''}) \tau_{kp} \tau_{kp} \right) \sinh J v_p \right\} + \frac{i}{2} (\sigma_k, \sigma_{k'}) \left\{ [t_{kk}, \tau_{kk}] - \\
- [t_{k', k''}, \tau_{kk}] + [t_{k', k''} + [t_{kk}, \tau_{kk}]] \right\} \cosh J v_k + \frac{1}{2} \left\{ t_{kk}^2 - \\
- t_{k' k''}^2 + [t_{k', k''}, t_{kk}] + (\sigma_k, \sigma_{k'})^2 \left( \tau_{kk}^2 - \tau_{k' k''}^2 - [\tau_{k' k''}, \tau_{kk}] \right) \right\} \sinh J v_k, \\
\frac{1}{2} \sinh J v_k = i(\sigma_k, \sigma_{n+1})(\sigma_{n+1}, \sigma_{k'}) \tau_{k,n+1} t_{k,n+1} + \\
+ 2 \sum_{p=1, p \neq k}^{n} \left\{ i \left( (\sigma_{k'}, \sigma_p)(\sigma_k, \sigma_p) t_{kp} \tau_{kp} + \\
+ (\sigma_k, \sigma_p)(\sigma_{k'}, \sigma_p) \tau_{kp} t_{kp} \right) \cosh J v_p + \left( (\sigma_{k'}, \sigma_p)(\sigma_k, \sigma_p) t_{kp} t_{kp} - \\
- (\sigma_k, \sigma_p)(\sigma_{k'}, \sigma_{k''}) \tau_{kp} \tau_{kp} \right) \sinh J v_p \right\} + \frac{i}{2} (\sigma_k, \sigma_{k'}) \left\{ [t_{kk}, \tau_{kk}] - \\
- [t_{k', k''}, \tau_{kk}] + [t_{k', k''} + [t_{kk}, \tau_{kk}]] \right\} \cosh J v_k + \frac{1}{2} \left\{ t_{kk}^2 - \\
- t_{k' k''}^2 + [t_{k', k''}, t_{kk}] + (\sigma_k, \sigma_{k'})^2 \left( \tau_{kk}^2 - \tau_{k' k''}^2 - [\tau_{k' k''}, \tau_{kk}] \right) \right\} \sinh J v_k, \\
0 = (\sigma_i, \sigma_{n+1})(\sigma_k, \sigma_{n+1}) t_{i,n+1} t_{k,n+1} + \\
+ 2 \sum_{p=1, p \neq i,k}^{n} \left\{ ((\sigma_i, \sigma_p)(\sigma_k, \sigma_p) t_{ip} t_{kp} + (\sigma_i, \sigma_{p'})(\sigma_k, \sigma_{p'}) \tau_{ip} \tau_{kp}) \cosh J v_p + \\
+ i ((\sigma_i, \sigma_p)(\sigma_k, \sigma_{p'}) t_{ip} \tau_{kp} - (\sigma_k, \sigma_p)(\sigma_i, \sigma_{p'}) \tau_{ip} t_{kp}) \sinh J v_p \right\} + \\
+ \left\{ (\sigma_i, \sigma_p)(\sigma_{k'}, \sigma_{p'}) t_{k'p} \tau_{kp} + (\sigma_k, \sigma_{p'}) (\sigma_i, \sigma_{p'}) (\tau_{k'p} + \tau_{kp}) \right\} \cosh J v_k + \\
+ \left\{ (\sigma_k, \sigma_{p'}) (\sigma_{k'}, \sigma_{p'}) (\tau_{k'k} + \tau_{kk}) \right\} \sinh J v_k + \\
+ \left\{ (\sigma_i, \sigma_{p'}) (\sigma_{k'}, \sigma_{p'}) (\tau_{k'k} + \tau_{kk}) \right\} \cosh J v_k + \\
+ \left\{ (\sigma_i, \sigma_{p'}) (\sigma_{k'}, \sigma_{p'}) (\tau_{k'k} + \tau_{kk}) \right\} \sinh J v_k,
\[ 0 = (\sigma_{t'}, \sigma_{n+1})(\sigma_{k'}, \sigma_{n+1})\tau_{i,n+1}\tau_{k,n+1} + \\
+ 2 \sum_{p=1, p \neq i,k}^n \left\{ \left( (\sigma_{t'}, \sigma_{p})(\sigma_{k'}, \sigma_{p})t'_{ip}t'_kp + (\sigma_{t'}, \sigma_{p})(\sigma_{k'}, \sigma_{p})\tau'_{ip}\tau'_kp \right) \cosh Jv_p + \right. \\
+ i \left( (\sigma_{t'}, \sigma_{p})(\sigma_{k'}, \sigma_{p})t'_{ip}\tau'_{kp} + (\sigma_{t'}, \sigma_{p})(\sigma_{k'}, \sigma_{p})\tau'_ip\tau'_{kp} \right) \sinh Jv_p \right\} + \\
+ \left\{ (\sigma_{t'}, \sigma_{p})(t_{ii} - t_{i'i})t'_kp + (\sigma_{t'}, \sigma_{p})(\tau_{ii} - \tau_{i'i})t'_kp \right\} \cosh Jv_i + \\
+ i \left\{ (\sigma_{t'}, \sigma_{p})(t_{ii} - t_{i'i})\tau'_{kp} + (\sigma_{t'}, \sigma_{p})(\tau_{ii} - \tau_{i'i})\tau'_kp \right\} \sinh Jv_i + \\
+ \left\{ (\sigma_{t'}, \sigma_{k'})(\tau'_{ik}(t_{kk} - t_{k'k}) + (\sigma_{t'}, \sigma_{k'})(\tau'_{ik}(\tau_{kk} - t_{k'k}) \right\} \cosh Jv_k + \\
+ i \left\{ (\sigma_{t'}, \sigma_{k'})(\tau'_{ik}(t_{kk} - t_{k'k}) + (\sigma_{t'}, \sigma_{k'})(\tau'_{ik}(\tau_{kk} - t_{k'k}) \right\} \sinh Jv_k, \\
\]
\[
0 = \sum_{p=1, p \neq k}^{n} \left\{ -i \left( (\sigma_{k}', \sigma_{n+1}) t_{n+1,k} t_{k+k} \right) \cosh J v_{k} \right. \\
+ \left. \left( (\sigma_{k}', \sigma_{p}) (\sigma_{p}', \sigma_{n+1}) t_{n+1,p} \tau_{k} \right) \cosh J v_{p} + i \left( (\sigma_{k}, \sigma_{p}) (\sigma_{p}, \sigma_{n+1}) t_{n+1,p} \tau_{k} \right) \sinh \cosh J v_{p} \right\},
\]

\[
0 = \sum_{p=1, p \neq k}^{n} \left\{ -i \left( (\sigma_{k}', \sigma_{p}) (\sigma_{p}, \sigma_{n+1}) t_{n+1,k} t_{k+k} \right) \cosh J v_{k} \right. \\
+ \left. \left( (\sigma_{k}, \sigma_{p}') (\sigma_{p}', \sigma_{n+1}) t_{n+1,p} \tau_{k} \right) \cosh J v_{p} + i \left( (\sigma_{k}', \sigma_{p}) (\sigma_{p}, \sigma_{n+1}) t_{n+1,p} \tau_{k} \right) \sinh \cosh J v_{p} \right\},
\]

\[
0 = -\sum_{p=1, p \neq k}^{n} \left\{ -i \left( (\sigma_{k}', \sigma_{p}) (\sigma_{p}, \sigma_{n+1}) t_{n+1,k} t_{k+k} \right) \cosh J v_{k} \right. \\
+ \left. \left( (\sigma_{k}, \sigma_{p}') (\sigma_{p}', \sigma_{n+1}) t_{n+1,p} \tau_{k} \right) \cosh J v_{p} + i \left( (\sigma_{k}', \sigma_{p}) (\sigma_{p}, \sigma_{n+1}) t_{n+1,p} \tau_{k} \right) \sinh \cosh J v_{p} \right\},
\]
\[ +i\left\{ (\sigma_k, \sigma_{k'}) (\sigma_k, \sigma_{n+1}) t_{n+1,k} (\tau_{kk} - \tau_{k'k}) - (\sigma_{k'}, \sigma_{n+1}) \tau_{n+1,k} (t_{kk} - t_{k'k}) \right\} \cosh J\psi_k. \]
References


